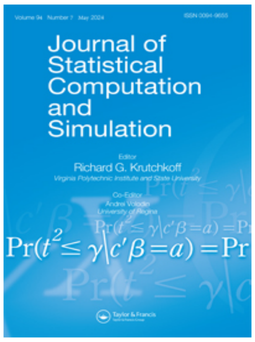


Journal of Statistical Computation and Simulation

ISSN: (Print) (Online) Journal homepage: www.tandfonline.com/journals/gscs20

Goodness of fit test for a skewed generalized normal distribution

Chengdi Lian, Ke Yang, Aidi Liu & Weihu Cheng

JOURNAL OF STATISTICAL COMPUTATION AND SIMULATION
https://doi.org/10.1080/00949655.2024.2349117

Goodness of fit test for a skewed generalized normal distribution

Chengdi Lian, Ke Yang, Aidi Liu and Weihu Cheng

School of Mathematics, Statistics and Mechanics, Beijing University of Technology, Beijing, People's Republic of China

ABSTRACT

The skewed generalized normal (SGN) distribution with four parameters is a versatile distribution that can effectively model data with skewness and heavy or light tails. In this paper, we conduct two classes of goodness of fit tests for the SGN distribution based on the empirical distribution function (edf) and the sample correlation coefficient. The first class involves transforming the sample into approximately mixed gamma observations, and then applying five classical parametric bootstrap edf-based goodness of fit tests. The second class is based on the inverse probability transformation and utilizes the sample correlation coefficient as the test statistic. We compare the finite sample performances of the proposed tests for different sample sizes and alternative distributions by extensive numerical studies. The simulation results demonstrate that the proposed tests provide a valid alternative to the standard tests using the original data, and the analysis of real data illustrates its application.

ARTICLE HISTORY
Received 15 September 2023
Accepted 24 April 2024

KEYWORDS
Goodness of fit tests; skewed generalized normal distribution; bootstrap; empirical distribution function

1. Introduction

With the advances in modern technologies, the application of skewed data has become widespread in various fields, such as biostatistics, economics, education, and sociology, among others. A typical example of such data is insurance risk data in finance and risk management, where extreme tails and skewness are commonly observed (see [1]). To model and analyse skewed data, existing literature has introduced various skewed distributions, including the three-parameter skew-normal (SN) (see [2–6]) and the four-parameter skewed generalized normal (SGN) (see [7–10]) distributions. Azzalini [11] noted that the SN distribution, due to its short tails, is not well-suited for situations requiring heavier tails than the normal distribution. In contrast, the SGN, featuring an additional shape parameter, tackles this problem by providing a broader range of skewness and enhancing its flexibility. Since the works of [12,13], among others, recent studies have further clarified the widespread application of the SGN model in diverse fields. For example, [14] effectively applied the SGN in modelling autoregressive (AR) time series processes, [15] used the SGN in income modelling to capture asymmetry, and [16] employed the SGN distribution to

CONTACT Weihu Cheng chengweihu@bjut.edu.cn School of Mathematics, Statistics and Mechanics, Beijing University of Technology, Pingleyuan NO.100, Chaoyang District, Beijing 100124, People's Republic of China

analyse temperature data. Due to the widespread application of SGN distributions in economics and environmental science, it is meaningful to develop a corresponding goodness of fit (GOF) test. Due to the widespread application of SGN distributions in economics and environmental science, it is meaningful to develop a corresponding goodness of fit (GOF) test.

Most of the current research is primarily focussed on the goodness of fit testing problem for the skewed normal (SN) distributions ([2–6]). One of the pioneer works in this area was done by Gupta and Chen [17]. They provided two conventional goodness of fit testing methods, the Kolmogorov-Smirnov test, and Pearson's χ^2 test. Mateu-Figueras et al. [18] proposed several GOF test statistics from the Kolmogorov-Smirnov and Cramer-von Mises families for the SN distribution, considering unknown location and scale parameters. Building upon the work of [18], subsequent researchers have developed many goodness of fit tests for the SN distribution. Rodríguez and Alva [19] provided two GOF tests for the SN distribution based on the quantile function and the sample correlation coefficients. The critical values were obtained using a parametric bootstrap method. Later, [20] introduced an empirical likelihood ratio test to assess skew normality and derived the asymptotic distributions of the test statistic under both the null and alternative hypotheses. To address the computational complexity, [21] developed a class of goodness of fit tests based on the characteristic function, which was applied to the skew-t distribution with known degrees of freedom. For extensions to the multivariate case, [22] proposed efficient test statistics based on the canonical form of the multivariate skewed normal distribution and derived the null distribution accordingly. More recently, [23] proposed a general parametric bootstrap edf-based goodness of fit tests for the sinh-arcsinh distribution. However, little research provides a comparative analysis of the GOF tests of SGN distributions due to the complexity of its cumulative distribution function, involving multiple parameters and special functions.

Although testing procedures for the SN family of distributions are well-established in the literature, to the best of our knowledge, there has been very little work on testing for the SGN distributions against other distributions. Inspired by the above observations, this paper focuses on developing goodness of fit tests for a novel SGN distribution. Lian et al. [24] proposed a novel skewed generalized t (SGT) distribution with parameters for location μ, scale σ, skewness r, and two shape parameters α, β, as a scale mixture of the SGN distribution. Notably, the SGN distribution is a special case of this SGT distribution proposed in [24] when the shape parameter α approaches infinity. The incorporation of the skewness and shape parameters enables the SGN distribution to possess some attractive properties, such as asymmetry and heavy tails, making it a widely used tool in the social and natural sciences. The SGN distribution exhibits excellent adaptability to a wide range of data due to its ability to accommodate a wide range of skewness and kurtosis, while allowing for independent adjustments of its location, scale, skewness, and shape parameters. Furthermore, the SGN distribution includes special cases like the Laplace, the normal, and the Pareto distributions, providing superior flexibility in modelling data with characteristics such as leptokurtosis, skewness, and variations in tail behaviour.

In this work, several goodness of fit tests are developed based on the properties of the SGN distribution. Specifically, we construct two classes of GOF tests for the SGN distribution with unknown parameters. The first involves transforming the data into approximately mixed gamma variables and applying five parametric bootstraps edf-based goodness of fit tests. The second is based on the inverse probability transformation and then uses sample correlation coefficients as the test statistic. We compare the five edf-based GOF tests based on the transformed samples with five classical edf-based GOF test statistics in various parameter settings, in terms of sizes and powers. The approximate distributions of the above test statistics are simulated based on a parametric bootstrap procedure. We investigate their performances against different types of alternative hypotheses by using Monte Carlo simulation. Simulation studies show that the Type I error can be well controlled for a given nominal level for the proposed test statistics, especially for moderate to large sample sizes. The power comparisons with the five classical edf-based tests using original data indicate that the proposed tests are very competitive candidates for the goodness of fit test of our SGN distribution. Notably, the developed tests are generally more powerful against symmetric alternatives than skewed alternatives.

This paper is organized as follows. In Section 2, we introduce a class of skewed generalized normal distributions that encompass many common distributions as special cases. An iterative algorithm is used to compute the maximum likelihood (ML) estimates of the parameters of the SGN distribution. In Section 3, we apply different goodness of fit test statistics and discuss the parametric bootstrap procedure. In Section 4, we explore the size and power characteristics of the proposed tests against diverse alternative distributions by simulation. In Section 5, we present a real example to illustrate the usefulness of the proposed procedure. Finally, Section 6 provides some conclusions and discussions.

2. Skewed generalized normal distributions

Before discussing the goodness of fit tests for the SGN distribution proposed by Lian et al. [24], we first give an overview of several types of SGN distributions. The skewed generalized normal distributions are the most popular models in economics, finance and related areas. The literature has introduced several SGN distributions, employing different construction methods. For example, [25] proposed a two-piece generalized skew-normal (GSN) distribution based on a similar extension proposed by Azzalini [26]. Its probability density is given by

$$f_{GSN}(x \mid \mu, \sigma, \alpha, \beta) = \begin{cases} \dfrac{2}{\sigma}\phi\left(\dfrac{x-\mu}{\sigma(1+\beta)}\right)\left[\dfrac{\beta}{1+\beta} + \dfrac{(1-\beta)}{1+\beta}\Phi\left(\dfrac{\alpha(x-\mu)}{\sigma(1+\beta)}\right)\right], & \text{if } x < \mu, \\ \dfrac{2}{\sigma}\phi\left(\dfrac{x-\mu}{\sigma(1-\beta)}\right)\Phi\left(\dfrac{\alpha(x-\mu)}{\sigma(1-\beta)}\right), & \text{if } x \geq \mu, \end{cases}$$

where $\mu, \alpha \in \mathbb{R}, \beta \in [0,1), \sigma > 0, \phi(\cdot)$ and $\Phi(\cdot)$ denote, respectively, the density and distribution functions of the standard normal distribution. The GSN distribution is a mixture of a normal distribution and a dependent discrete distribution. In the special case where $\beta = 0$, it corresponds to the SN distribution proposed by Azzalini [27].

Bekker et al. [10] introduced a location-scale SGN distribution by applying a skewing method to a generalized normal distribution. The SGN distribution of [10] has a density function defined as follows:

$$f(x; \mu, \alpha, \beta, \lambda) = \frac{2}{\alpha}\phi^*\left(\frac{x-\mu}{\alpha}; \beta\right)\Phi\left(\sqrt{2}\lambda\left(\frac{x-\mu}{\alpha}\right)\right), \quad x \in \mathbb{R},$$

where $\phi^*(y;\beta) = \frac{\beta}{2\alpha\Gamma(1/\beta)}\exp\{-|y|^\beta\}$, the location parameter $\mu \in \mathbb{R}$, scale parameter $\alpha \in \mathbb{R}^+$, shape parameter $\beta \in \mathbb{R}^+$ and skewness parameter $\lambda \in \mathbb{R}$. If $\mu = 0, \alpha = \sqrt{2}$ and $\beta = 2$, the SGN distribution of [10] reduces to that of Azzalini's SN. Azzalini [27].

More recently, [24] introduces a novel skewed generalized t (SGT) distribution, which has five parameters, and the SGN distribution is a special case of the SGT distribution when the shape parameter α approaches infinity. In this paper, we restrict our attention to this SGN proposed by Lian et al. [24] because of its good transformation properties. As studied in [24], the SGN distribution has a highly flexible shape, which makes it suitable for fitting a wide range of data. The probability density function (pdf) of a random variable X following our SGN distribution is defined as follows:

$$f_{SGN}(x;\mu,\sigma,r,b) = \frac{b}{2^{1+1/b}\Gamma(1/b)\sigma}\exp\left\{-\frac{|x-\mu|^b}{2\sigma^b[1+r\cdot\text{sign}(x-\mu)]^b}\right\},\quad x\in R,\quad (1)$$

with the location parameter $\mu \in R$, the scale parameter $\sigma > 0$, the skewness parameter $|r| \le 1$, the shape parameter $b > 0$, and $\Gamma(\cdot)$ denoting the gamma function. For the SGN variable X, we denote by $X \sim \text{SGN}(\mu,\sigma,r,b)$. As shown in Figure 1, the density function of SGN is continuous and unimodal with the mode at the centre $x = \mu$. This unimodal property is evident from the expression of the pdf in Equation (1). The density curve is skewed to the left for negative values of r and skewed to the right for positive values of r. When reversing the sign of r, the density mirrors on the opposite side of the vertical axis, increasing skewness with the value of $|r|$. The shape parameter b controls the tails and peak shapes of the density curve. Smaller values of b result in a heavier tail and sharper peak. Density curves with extremely sharp peaks ($b < 1$) are uncommon and are not considered in the current study. Let $X_0 = (X-\mu)/\sigma$ denote a standardized SGN variable, then the cumulative distribution function (cdf) of X_0 is given by

$$F_{X_0}(x;r,b) = \begin{cases} \frac{1-r}{2}\left[1-\frac{\gamma(u(x;r,b);1/b)}{\Gamma(1/b)}\right], & x \le 0, \\ \frac{1-r}{2}+\frac{1+r}{2}\cdot\frac{\gamma(u(x;r,b);1/b)}{\Gamma(1/b)}, & x > 0, \end{cases} \quad (2)$$

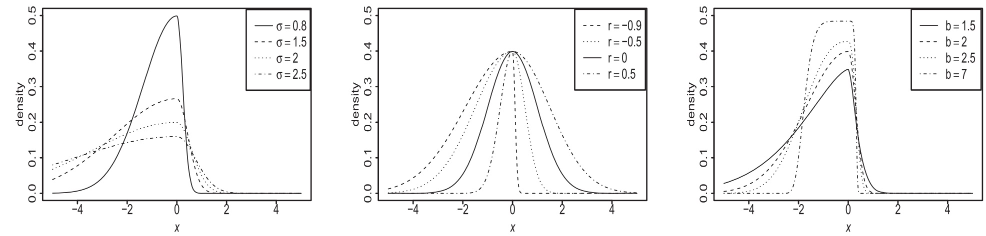
where $u(x;r,a,b) = |x|^b/\{2\left[1+r\cdot\text{sign}(x)\right]^b\}$ and $\gamma(y;a) = \int_0^y t^{a-1}e^{-t}dt$ denotes the lower incomplete gamma function. It follows that $F_X(x;\mu,\sigma,r,b) = F_{X_0}((x-\mu)/\sigma;r,b)$.

We first give a brief review of the stochastic representation of the SGN distribution, which is particularly important in generating random samples and deriving its main properties. Then, we investigate other crucial properties of our SGN distribution in this section, which play an important role in implementing of goodness of fit tests for the SGN distribution.

Proposition 2.1: *Let $X \sim \text{SGN}(\mu,\sigma,r,b)$. Then, the stochastic representation of X is given by*

$$X \overset{d}{=} \mu + \sigma 2^{1/b} W Y^{1/b}, \quad (3)$$

where $P(W = r+1) = (r+1)/2$, $P(W = r-1) = (1-r)/2$, $Y \sim Ga(1/b, 1)$, W and Y are independent, and $Ga(a,b)$ denotes the gamma distribution with shape and scale parameters a and b.



(a) Pdf of SGN for different values of σ with $\mu = 0, r = -0.7, b = 2$.
(b) Pdf of SGN for different values of r with $\mu = 0, \sigma = 1, b = 2$.
(c) Pdf of SGN for different values of b with $\mu = 0, \sigma = 1, r = -0.7$.

Figure 1. Densities of the SGN distributions for different parameter values. (a) Pdf of SGN for different values of σ with $\mu = 0, r = -0.7, b = 2$. (b) Pdf of SGN for different values of r with $\mu = 0, \sigma = 1, b = 2$ and Pdf of SGN for different values of b with $\mu = 0, \sigma = 1, r = -0.7$.

The kth moments of X are given by

$$E(X^k) = \sum_{j=0}^{k} \binom{k}{j} \sigma^j \mu^{k-j} E(X_0^j), \quad k = 1, 2, \ldots \tag{4}$$

where $E(X_0^j)$ is given by

$$E(X_0^j) = 2^{j/b-1} \frac{\Gamma((j+1)/b)}{\Gamma(1/b)} \left[(r+1)^{j+1} - (r-1)^{j+1}\right].$$

In particular, the mean and variance of X are given by

$$E(X; \mu, \sigma, r, b) = \mu + \sigma r \frac{2^{1/b+1}\Gamma(2/b)}{\Gamma(1/b)},$$

$$Var(X; \sigma, r, b) = \frac{\sigma^2 2^{2/b}}{\Gamma(1/b)} \left[(3r^2+1)\Gamma(3/b) - \frac{4r^2\Gamma^2(2/b)}{\Gamma(1/b)}\right].$$

The analytical expressions for the moments can be easily verified from Proposition 2.1. Then we establish the following propositions that give two important transformations for constructing the goodness of fit tests for SGN distribution.

Proposition 2.2: *Let* $X \sim \mathrm{SGN}(\mu, \sigma, r, b), T = |X - \mu|^b / \sigma^b$, *then the following results hold:*

(1) *As* $r \neq 0, \pm 1$, *the pdf and cdf of the random variable* T *are given by*

$$f_T(t; r, b) = \frac{t^{1/b-1}}{2^{1+1/b}\Gamma(1/b)} \left[\mathrm{e}^{-\frac{t}{2(1-r)^b}} + \mathrm{e}^{-\frac{t}{2(1+r)^b}}\right], \quad t \geq 0, \tag{5}$$

and

$$F_T(t; r, b) = \frac{1-r}{2} \cdot \gamma\left(t; \frac{1}{b}, \lambda_1\right) + \frac{1+r}{2} \cdot \gamma\left(t; \frac{1}{b}, \lambda_2\right), \quad t \geq 0, \tag{6}$$

respectively, where $\lambda_1 = \frac{1}{2(1-r)^b}$, $\lambda_2 = \frac{1}{2(1+r)^b}$ *and* $\gamma(x; \alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^x t^{\alpha-1} e^{-\lambda t} \mathrm{d}t$ *denotes the lower incomplete gamma function;*

(2) *As* $r = 0$, *we have* $T \sim Ga(1/b, 2)$;
(3) *As* $r = \pm 1$, *we have* $T \sim Ga(1/b, 2^{1+1/b})$.

Proof: See Appendix A.1. ■

Proposition 2.3: *Let* $X \sim \mathrm{SGN}(\mu, \sigma, r, b)$, $Y = (X - \mu)/\sigma$. *Then* $Y \sim \mathrm{SGN}(0, 1, r, b)$ *and the inverse cdf of* Y *can be given by*

$$F_Y^{-1}(u; r, b) = \begin{cases} -(1-r) \cdot \left[2\gamma^{-1}\left(1 - \frac{2u}{1-r}; \frac{1}{b}, 1\right)\right]^{1/b}, & u \le \frac{1-r}{2} \\ (1+r) \cdot \left[2\gamma^{-1}\left(\frac{2}{1+r}\left(u - \frac{1-r}{2}\right); \frac{1}{b}, 1\right)\right]^{1/b}, & u > \frac{1-r}{2}, \end{cases} \tag{7}$$

where $\gamma^{-1}(\cdot; p, q)$ *is the inverse of the lower incomplete gamma function defined in Proposition* 2.2.

Proof: This result can be obtained by straightforward algebraic manipulations based on Equation (2). ■

2.1. Maximum likelihood estimation for the SGN distribution

To obtain the test statistics, we first need to estimate the unknown parameters. To solve the ML estimator of the parameter of the SGN distribution, we propose a new iterative algorithm to maximize the likelihood function. This algorithm overcomes the difficulty of directly solving the estimation equations.

Let $\mathcal{F} = \{F_X(\cdot; \boldsymbol{\theta}) : \boldsymbol{\theta} \in \Theta\}$ be the set of the distribution functions for univariate SGN, where $\boldsymbol{\theta} = (\mu, \sigma, r, b)^\top$ denotes the parameter vector. Let $\mathbf{X}_n = \{X_1, \ldots, X_n\}$ be an independent and identically distributed (i.i.d.) random sample drawn from a member of $\mathcal{F}$ with the parameter $\boldsymbol{\theta}$. Thus, the log-likelihood function of $\boldsymbol{\theta}$, given the observed data $\mathbf{X}_n$, is

$$\ell(\boldsymbol{\theta}|\mathbf{X}_n) = \sum_{i=1}^{n} \left\{ \ln b - (1 + 1/b)\ln 2 - \ln\Gamma(1/b) - \ln\sigma - \frac{|X_i - \mu|^b}{2\sigma^b[1 + r \cdot \mathrm{sign}(X_i - \mu)]^b} \right\}. \tag{8}$$

When $\boldsymbol{\theta}$ is unknown, we use the ML method. However, due to the complexity of Equation (8), which contains non-differentiable and nonlinear functions like $\mathrm{sign}(\cdot)$ and $\psi(\cdot)$, it is difficult to obtain the ML estimate using traditional optimization techniques relying on derivatives. In order to maximize the log-likelihood $\ell(\boldsymbol{\theta}|\mathbf{X}_n)$, we take the derivatives of this log-likelihood function with respect to the parameters μ, σ, r and b at the $(h+1)$ iteration step. To ensure fast convergence of the algorithm, we utilize different numerical optimization methods to update $\widehat{\boldsymbol{\theta}}^{(h+1)}$. This iterative algorithm follows these steps:

By setting the derivative of the the log-likelihood $\ell(\boldsymbol{\theta}|\mathbf{X}_n)$ with respect to μ to zero at the $(h+1)$ iteration step, we have

$$\sum_{i=1}^{n} \frac{\text{sign}\,(X_i-\mu)\,|X_i-\mu|^{b-1}}{\left[1+r\cdot \text{sign}\,(X_i-\mu)\right]^{b}} = 0. \tag{9}$$

Let $\rho(y) = \frac{\text{sign}(y)|y|^{b-1}}{[1+r\cdot\text{sign}(y)]^{b}}$, the estimate of μ can be defined by an implicit equation: $\sum_{i=1}^{n}\rho(X_i-\mu)=0$. According to the method proposed in [28,29], a numerical solution of μ is given as a weighted mean:

$$\mu = \frac{\sum_{i=1}^{n} w_i X_i}{\sum_{i=1}^{n} w_i}, \quad \text{where } w_i = \frac{\rho(X_i-\mu)}{X_i-\mu}. \tag{10}$$

We apply (10), and the solution of Equation (9) yields the solution of μ at the $(h+1)$ step:

$$\hat{\mu}^{(h+1)} = \frac{\sum_{i=1}^{n} w_i(X_i;\hat{\mu}^{(h)},\hat{\sigma}^{(h)},\hat{r}^{(h)},\hat{b}^{(h)})X_i}{\sum_{j=1}^{n} w_j(x_j;\hat{\mu}^{(h)},\hat{\sigma}^{(h)},\hat{r}^{(h)},\hat{b}^{(h)})}, \tag{11}$$

where $w_i(X_i;\mu,\sigma,r,b) = \frac{|X_i-\mu|^{b-2}}{[1+r\cdot\text{sign}(X_i-\mu)]^{b}}$.

We consider the derivative of Equation (8) with the scale parameter σ at the $(h+1)$ iteration step as follows

$$\frac{2n\sigma^{b}}{b} - \sum_{i=1}^{n} \frac{|X_i-\mu|^{b}}{\left[1+r\cdot\text{sign}\,(X_i-\mu)\right]^{b}} = 0. \tag{12}$$

Fixing $\mu=\hat{\mu}^{(h+1)}$, we update $\hat{\sigma}^{(h+1)}$ by the following closed-form expression

$$\hat{\sigma}^{(h+1)} = \left\{\frac{\hat{b}^{(h)}}{2n}\sum_{i=1}^{n} w_i\left(X_i;\hat{\mu}^{(h+1)},\hat{\sigma}^{(h)},\hat{r}^{(h)},\hat{b}^{(h)}\right)\left(X_i-\hat{\mu}^{(h+1)}\right)^2\right\}^{\frac{1}{\hat{b}^{(h)}}}. \tag{13}$$

Similarly, setting $\partial\ell(\boldsymbol{\theta}|\mathbf{X}_n)/\partial r$ to zero, we have

$$\sum_{i=1}^{n} \frac{\text{sign}\,(X_i-\mu)\,|X_i-\mu|^{b}}{\left[1+r\cdot\text{sign}\,(X_i-\mu)\right]^{b+1}} = 0. \tag{14}$$

Fix $\mu=\hat{\mu}^{(h+1)}$ and $\sigma=\hat{\sigma}^{(h+1)}$, and update $\hat{r}^{(h+1)}$ as the solution of Equation (14)

$$\hat{r}^{(h+1)} = 1-2\left\{\left[\frac{\sum_{i=1}^{n}\gamma_{i1}(X_i;\hat{\mu}^{(h+1)})\left(\left[X_i-\hat{\mu}^{(h+1)}\right]^{+}\right)^{\hat{b}^{(h)}}}{\sum_{i=1}^{n}\gamma_{i2}(X_i;\hat{\mu}^{(h+1)})\left(\left[\hat{\mu}^{(h+1)}-X_i\right]^{+}\right)^{\hat{b}^{(h)}}}\right]^{\frac{1}{\hat{b}^{(h)}+1}}+1\right\}^{-1}, \tag{15}$$

where $\gamma_{i1}(X_i;\mu) = I(X_i \geq \mu)$, $\gamma_{i2}(X_i;\mu) = 1-\gamma_{i1}(X_i;\mu) = I(X_i<\mu)$ and $[x]^{+} = \max\{x,0\}$.

Next, we update the estimate of the shape parameter b. This involves fixing the values of μ, σ, r to the values of $(h+1)$th step, and updating the estimate of b using the Newton-Raphson method ([30,31]). Each iteration requires the first and second derivatives of the objective function $\ell(\boldsymbol{\theta}|\mathbf{X}_n)$ with respect to the parameter b

$$\hat{b}^{(h+1)} = \hat{b}^{(h)} - \frac{\partial \ell(\boldsymbol{\theta}|\mathbf{X}_n)}{\partial b} \left(\frac{\partial^2 \ell(\boldsymbol{\theta}|\mathbf{X}_n)}{\partial b^2} \right)^{-1} \Bigg|_{b=\hat{b}^{(h)}} . \tag{16}$$

In Equation (16), $\frac{\partial \ell(\boldsymbol{\theta}|\mathbf{X}_n)}{\partial b}$ and $\frac{\partial^2 \ell(\boldsymbol{\theta}|\mathbf{X}_n)}{\partial b^2}$ are given by

$$\begin{aligned}
\frac{\partial \ell(\boldsymbol{\theta}|\mathbf{X}_n)}{\partial b} &= n\left[\frac{1}{b} + \frac{\ln 2}{b^2} + \frac{\psi\left(\frac{1}{b}\right)}{b^2}\right] - \frac{1}{2}\sum_{i=1}^{n}\left[h\left(X_i;\widehat{\boldsymbol{\theta}}^{*(h)}\right)\right]^b \ln h\left(X_i;\widehat{\boldsymbol{\theta}}^{*(h)}\right), \\
\frac{\partial^2 \ell(\boldsymbol{\theta}|\mathbf{X}_n)}{\partial b^2} &= -n\left[\frac{1}{b^2} + \frac{2\ln 2}{b^3} + \frac{2}{b^3}\psi\left(\frac{1}{b}\right) + \frac{1}{b^4}\psi'\left(\frac{1}{b}\right)\right] \\
&\quad - \frac{1}{2}\sum_{i=1}^{n}\left\{\left[h\left(X_i;\widehat{\boldsymbol{\theta}}^{*(h)}\right)\right]^b\left[\ln h\left(X_i;\widehat{\boldsymbol{\theta}}^{*(h)}\right)\right]^2\right\},
\end{aligned} \tag{17}$$

where $\widehat{\boldsymbol{\theta}}^{*(h)} = (\hat{\mu}^{(h+1)}, \hat{\sigma}^{(h+1)}, \hat{r}^{(h+1)}, b)^\top$, $h(X_i;\boldsymbol{\theta}) = \frac{|X_i-\mu|}{\sigma[1+r\cdot\text{sign}(X_i-\mu)]}$, $\psi(\cdot) = \Gamma'(\cdot)/\Gamma(\cdot)$ and $\psi'(\cdot)$ denote the digamma and trigamma functions, respectively.

The algorithm iterates until some suitable convergence criteria are satisfied. In this paper, the algorithm is terminated when the relative change in the log-likelihood function defined in Equation (8) is less than the per-specified tolerance 10^{-7}, i.e. $|\ell(\widehat{\boldsymbol{\theta}}^{(h+1)}|\mathbf{X}_n)/\ell(\widehat{\boldsymbol{\theta}}^{(h)}|\mathbf{X}_n) - 1| < 10^{-7}$ ([32]). This iterative process ensures that the estimated parameter progressively converge towards the maximum likelihood estimate. Each parameter update method is selected based on its effectiveness in maximizing the likelihood equations, leading to an efficient and reliable iterative algorithm.

This algorithm is summarized in Algorithm 1.

Algorithm 1: A new algorithm for solving the ML estimator of the SGN distribution.

Input :
A sample $\mathbf{X}_n$;
An initial estimator $\widehat{\boldsymbol{\theta}}^{(0)} = (\hat{\mu}^{(0)}, \hat{\sigma}^{(0)}, \hat{r}^{(0)}, \hat{b}^{(0)})^\top$;

Output:
$\widehat{\boldsymbol{\theta}}_{MLE}$;

```
1 Set h = 0; repeat
2     Update the location parameter μ̂^(h+1) by using (11);
3     Update the scale parameter σ̂^(h+1) by using (13);
4     Update the skewness parameter r̂^(h+1) by using (15);
5     Update the shape parameter b̂^(h+1) by using (16).
6 until |ℓ(θ̂^(h+1)|X_n)/ℓ(θ̂^(h)|X_n) − 1| < 10^−7;
```

Initialization:

It is well known that a suitable initial value can accelerate the convergence of the optimization algorithm. Note that μ is the mode of the SGN distribution. Thus, a robust estimator $\hat{\mu}^{(0)}$ can be obtained using the half range mode (HRM) method [33]. The HRM estimator is available through the *half.range.mode* function in the *genefilter* package or the *hrm* function in the *modeest* package in R software. For a detailed description of the algorithm, please refer to [24]. In our study, we use the following formulas to determine the initial values for $\hat{\mu}^{(0)}$, $\hat{\sigma}^{(0)}$ and $\hat{r}^{(0)}$

$$\hat{\mu}^{(0)} = \text{mode}(\mathbf{X}_n), \quad \hat{\sigma}^{(0)} = \sqrt{\frac{\sum_{i=1}^{n}(X_i - \hat{\mu}^{(0)})^2}{n}}, \quad \hat{r}^{(0)} = 1 - \frac{2\sum_{i=1}^{n} I(X_i \leq \mu)}{n}. \tag{18}$$

In symmetric or slightly skewed data situations, the sample mean and sample median can also offer a reasonable estimate for the location parameter μ. To determine the initial value $\hat{b}^{(0)}$:

(1) Start by selecting a suitable interval for the shape parameter b. Generate a series of b values within this interval with appropriate spacing and denote the qth element of the vector as $\hat{b}_q^{(0)} (q = 1, 2, \ldots)$.
(2) Calculate the values of the likelihood function, defined by Equation (8), for each $\widehat{\boldsymbol{\theta}}_{(q)} = (\hat{\mu}^{(0)}, \hat{\sigma}^{(0)}, \hat{r}^{(0)}, \hat{b}_q^{(0)})^\top$.
(3) The initial value of b is chosen as $\hat{b}_q^{(0)}$ which corresponds to the maximum value of the likelihood function.

Finally, we recommend trying multiple starting points and comparing their log-likelihood values in order to identify the true ML estimates as suggested in [34].

2.2. Consistency of the ML estimators and the convergence of the algorithm

In this subsection, we establish consistency of the ML estimates of all parameters of the SGN distribution and analyse the convergence of the ML algorithm. Theorem 2.1 establishes the consistency of the ML estimates for all parameters of the SGN distribution. The proof follows the framework for proving the consistency of ML estimates of unimodal densities proposed by Reiss [35] and Bryant and Williamson [36].

Theorem 2.1: *Let $\mathbf{X}_n = \{X_1, X_2, \ldots, X_n\}$ be an i.i.d. random sample from a member of $\mathcal{F}$ with a parameter $\boldsymbol{\theta} \in \Theta = \{(\mu, \sigma, r, b) : \mu \in \mathbb{R}, \sigma \in \mathbb{R}^+, r \in (-1, 1), b > 1 \text{ and } 1/b \in \mathbb{I}\}$, where $\mathbb{I}$ denotes any bounded closed interval away from 0. If $\boldsymbol{\theta} = \boldsymbol{\theta}_0 = (\mu_0, \sigma_0, r_0, b_0)^\top$, the ML estimator $\widehat{\boldsymbol{\theta}}_{MLE}$ for the SGN distribution is strongly consistent in the sense*

$$P\left(\lim_{n\to\infty} \|\widehat{\boldsymbol{\theta}}_{MLE} - \boldsymbol{\theta}_0\| = 0\right) = 1, \tag{19}$$

where $\| \bullet \|$ denotes Eucidian distance between the two sets $\mathscr{D}_1, \mathscr{D}_2$, which is given by

$$\|\mathscr{D}_1 - \mathscr{D}_2\| = \inf_{d_{1i}\in\mathscr{D}_1} \inf_{d_{2i}\in\mathscr{D}_2} \|d_{1i} - d_{2i}\|.$$

Proof: See Appendix A.2. ■

We now investigate the convergence of the Algorithm 1. The closed-form expression for the Fisher information matrix $I(\boldsymbol{\theta})$ for the ML estimates of the SGN parameters are given in Theorem 2.2. The proof and further details are provided in the Appendix A.3.

Theorem 2.2: *Let* $\Theta = \{(\mu, \sigma, r, b) : \mu \in \mathbb{R}, \sigma \in \mathbb{R}^+, r \in (-1, 1), b > 1 \text{ and } 1/b \in \mathbb{I}\}$, *where* $\mathbb{I}$ *is defined in Theorem* 2.1. *The elements of the Fisher information matrix, denoted by* I_{ij}, *are*

$$I_{ij} = -\mathrm{E}\left[\frac{\partial^2 \ell(\boldsymbol{\theta}|\mathbf{X}_n)}{\partial\theta_i \partial\theta_j}\right], \quad i, j = 1, 2, 3, 4, \tag{20}$$

where $I_{ij} = I_{ji}$ *and* θ_j *represents the jth element of the parameter vector* $\boldsymbol{\theta} = (\mu, \sigma, r, b)^\top$. *Then, we have*

$$I = \begin{pmatrix} I_{11} & 0 & I_{13} & 0 \\ 0 & I_{22} & 0 & I_{24} \\ I_{13} & 0 & I_{33} & 0 \\ 0 & I_{24} & 0 & I_{44} \end{pmatrix}, \tag{21}$$

where the corresponding components of Equation (21) *are listed in Appendix* A.3.

Proof: See Appendix A.3. ■

Numerical computations confirm the positive definiteness of the information matrix within the parameter space Θ. Thus, the proposed algorithm in Section 2.1 is convergent in the parameter space Θ.

3. Goodness of fit test

Let $\mathbf{Z}_n = \{Z_1, Z_2, \ldots, Z_n\}$ be a random sample with distribution function F_0, with support in $\mathbb{R}$ and finite mean. We are interested in testing the composite null hypothesis

$$H_0 : F_0 \in \mathcal{F} \quad \text{versus } H_1 : F_0 \notin \mathcal{F}, \tag{22}$$

where $\mathcal{F}$ is the SGN distribution functions with unknown parameter $\boldsymbol{\theta} \in \mathcal{R}^4$.

3.1. Traditional edf-based goodness of fit tests

In this subsection, we will apply five traditional edf-based goodness of fit test statistics. We first estimate the parameter under the null hypothesis H_0 using the maximum likelihood method; see Section 2.1. In the very unlikely situation that one or more parameters are known, some adjustments may need to be made to the following methods. Let

$$F_n(z) = \frac{1}{n}\sum_{i=1}^{n} I\left(Z_i \le z\right)$$

denotes the empirical distribution function of the random sample evaluated at x, where I denotes the indicator function, and $F_Z(\cdot; \widehat{\boldsymbol{\theta}})$ denotes the cdf of the best-fitted model in

$\mathcal{F}$, where $\widehat{\boldsymbol{\theta}}$ is the ML estimate of $\boldsymbol{\theta}$. In parametric edf-based goodness of fit testing, test statistics are used to measure the discrepancy between $F_n(z)$ and $F_Z(\cdot;\widehat{\boldsymbol{\theta}})$.

D'Agostino [37] and Zamanzade [38] provided a complete and detailed review on edf-based GOF tests. As introduced by Pewsey [23] and Cabras and Castellanos [39], the edf-based test statistics considered in this work are the Kolmogorov-Smirnov (D), the Kuiper (V), the Cramer-von Mises (W^2), the Watson (U^2) and the Anderson-Darling (A^2) statistics. It should be noted that A^2 statistic introduces a different weighting function to the distance $|F_n(z) - F_Z(\cdot;\widehat{\boldsymbol{\theta}})|$ compared to D, and can therefore be seen as a modification of D. In fact, A^2 gives a higher weight to the lower and upper parts of the underlying distribution [40,41].

Let $D^+ = \max_i\{\frac{i}{n} - \hat{U}_{(i)}\}$, $D^- = \max_i\{\hat{U}_{(i)} - \frac{i-1}{n}\}$, then the corresponding sample versions are given as follows:

$$D^* = \max\left(D^+, D^-\right), \quad V^* = D^+ + D^-, \tag{23}$$

$$W^{2*} = \frac{1}{12n} + \sum_{i=1}^{n}\left(\hat{U}_{(i)} - \frac{2i-1}{2n}\right)^2, \tag{24}$$

$$U^{2*} = W^{2*} - n(\bar{\hat{U}} - 1/2)^2, \tag{25}$$

$$A^{2*} = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\ln \hat{U}_{(i)} + \ln\left(1 - \hat{U}_{(n+1-i)}\right)\right], \tag{26}$$

where $\hat{U}_i = F_Z(Z_i;\widehat{\boldsymbol{\theta}}), i = 1,\dots,n$ denote the pseudouniform variates distributed on $[0,1]$, $\hat{U}_{(1)} \le \hat{U}_{(2)} \le \cdots \le \hat{U}_{(n)}$ denote their ordered statistics, $\bar{\hat{U}} = \sum_{i=1}^{n}\hat{U}_i/n$, and $Z_i, i = 1,\dots,n$ are the given data points. If $F_Z(\cdot;\widehat{\boldsymbol{\theta}})$ provides a close approximation to the true distribution F_0, then $\hat{U}_i$ will be asymptotically uniformly distributed on the interval $[0,1]$. Formulas (23)–(26) indicate that all five test statistics are designed to detect departures of $\hat{U}_i$ from uniformity on $[0,1]$. Therefore, for each statistic, we reject the null hypothesis if the value of the statistic is larger than the $100(1-\alpha)\%$ quantile of the corresponding null distribution. It should be noted that the values of the above test statistics are unaffected if the data are first standardized.

The limiting null distributions of edf-based GOF test statistics are usually untractable since the distribution theories depend on various factors, such as the distribution being tested, the method of estimation, the values of parameter estimates, and the sample size. To apply the goodness of fit tests in practice, we use the following parametric bootstrap to approximate the null distributions of the test statistics when all parameters of the SGN distribution are assumed to be unknown.

Let $\mathcal{T}$ denote one of the five test statistics defined in formulas (23)–(26). Then the bootstrap algorithm for testing hypothesis (22) can be given as follows:

(1) Given the sample $\mathbf{Z}_n = \{Z_1, Z_2, \dots, Z_n\}$,
   (a) (a)Calculate the ML estimate $\widehat{\boldsymbol{\theta}}$.
   (b) (b)Evaluate $\mathcal{T} = \mathcal{T}(\mathbf{Z}_n;\widehat{\boldsymbol{\theta}})$. Denote the value obtained by $\mathcal{T}_0$.

(2) Generate B bootstrap samples from $F_{SGN}(\cdot;\widehat{\boldsymbol{\theta}})$, denoted as $\mathbf{Z}_j^* = \{Z_{1j}^*, Z_{2j}^*, \dots, Z_{nj}^*\}$, $j = 1, 2, \dots, B$.

(3) For the jth bootstrap sample, $\mathbf{Z}_j^*, j = 1, 2, \ldots, B$:
 (a) (a)Calculate the ML estimate $\widehat{\boldsymbol{\theta}}_j^*$, and hence identify $F_{SGN}(\cdot; \widehat{\boldsymbol{\theta}}_j^*)$.
 (b) (b)Calculate the value of the bootstrap test statistic $\mathcal{T}_j^* = \mathcal{T}(\mathbf{Z}_j^*; \widehat{\boldsymbol{\theta}}_j^*)$.
(4) The p value of the test is estimated by

$$\widehat{p} = \frac{\#\left\{1 \le j \le B : \mathcal{T}_j^* \ge \mathcal{T}_0\right\}}{B}. \tag{27}$$

3.2. A novel goodness of fit test based on the mixed gamma transform

The test procedure developed in this subsection is based on Proposition 2.2. Let

$$|Z - \mu|^b / \sigma^b = T, \tag{28}$$

where $Z \sim \mathrm{SGN}(\mu, \sigma, r, b)$, then we have

$$T \sim \frac{1-r}{2}\Gamma\left(\frac{1}{b}, \lambda_1\right) + \frac{1+r}{2}\Gamma\left(\frac{1}{b}, \lambda_2\right),$$

where λ_1 and λ_2 are defined in Proposition 2.2. Under H_0, the transformed data: $T_i, i = 1, \ldots, n$ are asymptotically independent samples distributed with mixed gamma distributions. A consistent estimator for $P(T \le x)$ is the empirical distribution function given as

$$F_n^T(t) = \begin{cases} 0, & t < t_{(1)}, \\ i/n, & t_{(i)} \le t < t_{(i+1)}, \\ 1, & t_{(n)} < t, \end{cases}$$

where $t_{(1)} \le \cdots \le t_{(n)}$ are the ordered statistics of the t_i's, then

$$F_T(t; r, b) \approx F_n^T(t). \tag{29}$$

Under the null hypothesis stated in (22), Equation (29) is expected to hold. If we have a consistent estimator $\widehat{\boldsymbol{\theta}}$ for parameter $\boldsymbol{\theta}$, the relationship in formula (29) is still expected to hold. To perform the test procedures, we need to first estimate the four parameters. To obtain $\hat{\mu}, \hat{\sigma}, \hat{r}$ and $\hat{a}$, we use the maximum likelihood method on the sample $Z_1, \ldots, Z_n$ from $SGN(\mu, \sigma, r, b)$. The values of $t_1, \ldots, t_n$ can be calculated using Equation (28). The proposed approach is to transform the original random sample $Z_1, \ldots, Z_n$ into an approximately mixed gamma sample $T_1, \ldots, T_n$, and then test the hypothesis $H_0' : T_1, \ldots, T_n \sim \frac{1-r}{2}\Gamma(\frac{1}{b}, \lambda_1) + \frac{1+r}{2}\Gamma(\frac{1}{b}, \lambda_2)$ using the five edf-based GOF tests discussed in Section 3.1.

The test procedure is summarized as follows:

(1) Given the sample $Z_1, \ldots, Z_n$ from the SGN distribution,
 (a) (a)Compute the ML estimates $\widehat{\boldsymbol{\theta}} = (\hat{\mu}, \hat{\sigma}, \hat{r}, \hat{b})^\top$; see the discussion in Section 2.1;
 (b) (b)Transform using

$$T_i = \frac{|Z_i - \hat{\mu}|^{\hat{b}}}{\hat{\sigma}^{\hat{b}}}, \quad i = 1, \ldots, n;$$

(c) (c)Use $\{T_i\}_{i=1}^n$ as input to the test procedures stated in Section 3.1, and substitute T_i for Z_i, $i = 1, \ldots, n$. Calculate five test statistics, denoted by D, V, W^2, U^2 and A^2, respectively.

(2) Generate B bootstrap samples from $F_{SGN}(\cdot; \widehat{\boldsymbol{\theta}})$, denoted as $\mathbf{Z}_j^* = \{Z_{1j}^*, Z_{2j}^*, \ldots, Z_{nj}^*\}$, $j = 1, 2, \ldots, B$.

(3) Let $\mathcal{S}$ denote one of the five test statistics, D, V, W^2, U^2 and A^2, and $\mathcal{S}_0$ denote the value of $\mathcal{S}$. For the jth bootstrap sample, $\mathbf{Z}_j^*, j = 1, 2, \ldots, B$:

(a) (a)Calculate the ML estimate $\widehat{\boldsymbol{\theta}}_j^* = (\hat{\mu}_j^*, \hat{\sigma}_j^*, \hat{r}_j^*, \hat{b}_j^*)^\top$, and hence identify $F_{SGN}(\cdot; \widehat{\boldsymbol{\theta}}_j^*)$.

(b) (b)Transform using

$$T_{ij}^* = \frac{|Z_{ij}^* - \hat{\mu}_j^*|^{\hat{b}_j^*}}{(\hat{\sigma}_j^*)^{\hat{b}_j^*}}, \quad i = 1, \ldots, n;$$

(c) (c)Calculate the value of the bootstrap test statistic $\mathcal{S}_j^* = \mathcal{S}(\mathbf{T}_j^*; \widehat{\boldsymbol{\theta}}_j^*)$, where $\mathbf{T}_j^* = \{T_{1j}^*, T_{2j}^*, \ldots, T_{nj}^*\}$.

(4) The p value of the test is estimated by

$$\widehat{p} = \frac{\#\left\{1 \le j \le B : \mathcal{S}_j^* \ge \mathcal{S}_0\right\}}{B}. \tag{30}$$

3.3. Goodness of fit test based on the sample correlation coefficient

The test procedure developed in this subsection is based on Proposition 2.3, which can be conducted as follows. Let $Z \sim \mathrm{SGN}(\mu, \sigma, r, b)$ with given μ and σ, then Z has a location and scale distribution,

$$P(Z \le z) = F_Y\left(\frac{z - \mu}{\sigma}; r, b\right), \tag{31}$$

where $F_Y(\cdot; r, b)$ is the distribution function of $Y \sim \mathrm{SGN}(0, 1, r, b)$, and it is given by Equation (2). Given the sample $Z_1, \ldots, Z_n$, the empirical distribution function $F_n(z)$ is a consistent estimator of $P(Z \le z)$, then

$$F_Y\left(\frac{Z_i - \mu}{\sigma}; r, b\right) \approx F_n(Z_i),$$

therefore

$$V_i := F_Y^{-1}\left(F_n(Z_i); r, b\right) \approx \frac{Z_i - \mu}{\sigma}. \tag{32}$$

We note that if Z has the SGN distribution, there is a strong linear relationship between the variables $V = F_Y^{-1}(F_n(Z); r, b)$ and Z under H_0. Based on this relation, we develop a GOF test for the SGN distribution with unknown parameters. It is worth noting that this linear relationship no longer holds for samples transformed in Section 3.2.

We define the sample correlation coefficient between V_i and Z_i,

$$r_n = \frac{\sum_{i=1}^{n}\left(Z_i - \bar{X}\right)\left(V_i - \bar{V}\right)}{\sqrt{\sum_{i=1}^{n}\left(Z_i - \bar{X}\right)^2 \sum_{i=1}^{n}\left(V_i - \bar{V}\right)^2}}, \tag{33}$$

as an estimator of the linear correlation between V and X. In practice, the critical values of the test can be approximated using B replicates of its bootstrap version ([42]). Bootstrap methods avoid the need for constructing and using tables of critical values. If μ, σ, r and b are estimated by a consistent estimator, say $\hat{\mu}, \hat{\sigma}, \hat{r}$ and $\hat{b}$, then it is expected that the linear relationship (32) still holds and a high value (approaching 1) of the r_n statistic will be observed. Notice that if the random sample comes from a distribution different from the SGN distribution, the formula (32) will not hold. We reject the null hypothesis H_0 at the level of significance α if $r_n \leq C_n(\alpha)$, where the critical value $C_n(\alpha)$ satisfies

$$\alpha = P\left(\text{Reject } H_0 \mid H_0\right) = P\left(r_n \leq C_n(\alpha)\right).$$

We use the following procedure to determine the critical values of r_n:

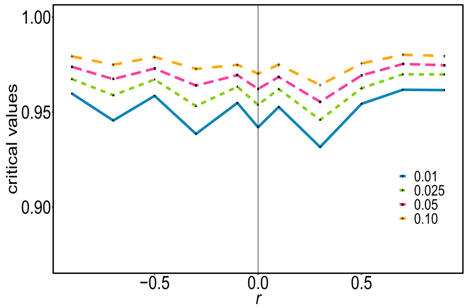
(1) Given a sample $\mathbf{Z}_n = \{Z_1, Z_2, \ldots, Z_n\}$ from $SGN(\mu, \sigma, r, b)$:
  (a) (a)Calculate the maximum likelihood estimator of the unknown parameter, denoted by $\widehat{\boldsymbol{\theta}} = (\hat{\mu}, \hat{\sigma}, \hat{r}, \hat{b})^\top$.
  (b) (b)Sort Z_i's into ascending order: $Z_{(1)} \leq \cdots \leq Z_{(n)}$.
  (c) (c)Calculate $v_{(i)} = F_Y^{-1}(F_n(Z_{(i)}); \hat{r}, \hat{b}), i = 1, \ldots, n$, where $F_Y^{-1}(\cdot; \hat{r}, \hat{b})$ is the quantile function given in formula (7).
  (d) (d)Evaluate r_n by substituting $Z_{(i)}, v_{(i)}$ generated in steps (b) and (c) into formula (33), denoted by r_{n0}.

(2) Generate B bootstrap samples from $F_{SGN}(\cdot; \hat{\mu}, \hat{\sigma}, \hat{r}, \hat{b})$, denoted by $\{\mathbf{Z}_j^* = \{Z_{1j}^*, Z_{2j}^*, \ldots, Z_{nj}^*\}, j = 1, 2, \ldots, B\}$.

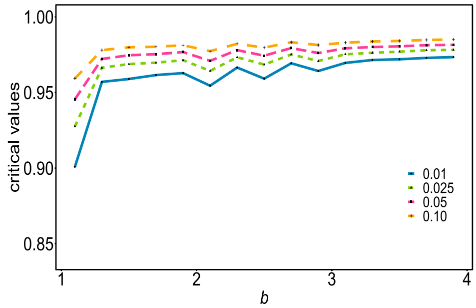
(3) For the jth bootstrap sample, $\mathbf{Z}_j^*, j = 1, 2, \ldots, B$, recalculate the ML estimate $\widehat{\boldsymbol{\theta}}_j^*$. Calculate the values of r_n, replacing Z_i by Z_{ij}^* and $\widehat{\boldsymbol{\theta}}$ by $\widehat{\boldsymbol{\theta}}_j^*$, denoted by $r_{nj}^*, j = 1, \ldots, B$.

(4) Reject H_0 if $r_{n0} \leq q_\alpha$, where q_α denotes the $100\alpha\%$ sample quantile of the r_{nj}^* values obtained in step 3.

It is relatively simple to find the critical values of r_n using the above procedures. Figure 2 presents graphs of the critical values for the r_n-based test as a function of r and b at significance levels $\alpha = \{0.01, 0.025, 0.05, 0.1\}$, using a bootstrap procedure with $B = 10{,}000$ Monte Carlo samples with sample size $n = 50$. It can be seen that the critical values of r_n become larger as the shape parameter b increases, indicating a tendency to reject the hypothesis that the data are from the SGN distribution. The critical values curve in Figure 2(a) is approximately symmetric about $r = 0$ when r is not around zero. Figure 2 shows that the distribution of the test statistic r_n under H_0 depends on the unknown parameters r and b.



(a) Critical values of r_n as a function of r. Samples are generated from SGN distribution with $\mu = 3,\ \sigma = 2$ and $b = 1.5$.



(b) Critical values of r_n as a function of b. Samples are generated from SGN distribution with $\mu = 3,\ \sigma = 2$ and $r = -0.5$.

Figure 2. Critical values of r_n as a function of skewness parameter r, and shape parameter b, obtained using parametric bootstrap when $\mathcal{F}$ is the SGN class. (a) Critical values of r_n as a function of r. Samples are generated from SGN distribution with $\mu = 3, \sigma = 2$ and $b = 1.5$ and (b) Critical values of r_n as a function of b. Samples are generated from SGN distribution with $\mu = 3, \sigma = 2$ and $r = -0.5$.

4. Simulation studies

In this section, we explore the finite sample performance of the proposed tests in different settings by simulations. Two pervasive features of any statistical test of interest are its rejection rate when H_0 is correct, i.e. the size, and when H_0 is false, i.e. the power. An attractive test is one whose size is sufficiently close to the nominal significance level and has high power.

The SGN distribution has four parameters, and in general, the values of all parameters are unknown. As discussed in Section 2.1, we can employ the iterative algorithm to obtain the maximum likelihood estimates. We then conducted five edf-based tests mentioned in Section 3.2 using the transformed sample, along with the test based on the sample correlation coefficient discussed in Section 3.3. We denote the resulting tests as D, V, W^2, U^2, A^2 and r_n, respectively. For comparative purposes, we also consider the standard test procedures in Section 3.1 based on the original sample, denoted as D^*, V^*, W^{2*}, U^{2*} and A^{2*}. In our study, we estimate the empirical sizes of the 11 tests at significance levels $\alpha = 0.01, 0.05$ and 0.10, with varying sample sizes $n = 20$, 50, 100, 200 and 500. Naturally, we expect that the proposed parametric bootstrap edf-based GOF tests, as described in Section 3, would effectively maintain the nominal significance level for small, medium, and large sample sizes. To examine the power of the proposed GOF tests, a Monte Carlo experiment was designed by varying the alternative distributions. By simulation, we explain the reliability of distinguishing between competing models solely based on the observed data in practice. In addition, we investigate the sensitivity of these tests to parameter estimates. For details, please refer to Appendix 2.

4.1. Size of the tests

In order to investigate the ability of the proposed test procedures to maintain a nominal significance level of 5%, we estimated the test sizes using 1000 pseudo-random samples of size n from the SGN distribution. We varied the skewness and shape parameters (r and b) of the SGN distribution while keeping $\mu = 3$ and $\sigma = 2$. Specifically, we set r values as 0, −0.2 and −0.7, denoting symmetric, slightly skewed to the left, and highly skewed to the left distributions, respectively. Additionally, we set the shape parameter b as 1.5, 2, and 2.5, representing distributions ranging from steep to flat in shape. Following the works of [23,42], for each (n, r, b) combination, we first generated 1000 samples of size n. For each sample, we simulated $B = 500$ parametric bootstrap samples of size n from the SGN distribution fitted to it using the maximum likelihood method. Then we applied the test procedures outlined in Section 3 at nominal significance levels of α. All computations were performed in the R program, utilizing the built-in '*uniroot*' routine for the maximum likelihood estimation.

Figure 3 shows the empirical sizes of the test statistics considered in this study for different sample sizes n under the nominal significance level of 5%. In each panel of Figure 3, the dashed horizontal line represents the limits of $0.05 \pm 1.96\sqrt{0.05(1-0.05)/1000}$ arising from normal approximation. To approximate the distributions of the test statistics, we employed a parametric bootstrap procedure described in Section 3 with $B = 500$ replicates. The results at significance levels of $\alpha = 0.1$ and 0.01 are given in Tables A8 and A9 in Appendix 2. Those lying outside the 95% confidence limits of $\alpha \pm 1.96\sqrt{\alpha(1-\alpha)/1000}$ have been marked as bold (below the lower limit) or as italic (above the upper limit). From Figure 3, it can be observed that most of the empirical size values that fall outside the limits are below the lower limit of such interval, indicating that the tests tend to be conservative in the corresponding scenarios. Additionally, the empirical sizes for $\alpha = 10\%$ and 1% exhibit similar trends, those for the 10% (1%) level being slightly more (less) conservative.

From Figure 3, we can observe that the tests based on the transformed sample, namely D, V, W^2, U^2, A^2 and r_n, can maintain the nominal significance level for most parameter combinations in large sample size $n = 200,\ 500$. Additionally, for small $n = 20$, 50, 100, the empirical sizes of the D, V, W^2, U^2, and A^2 tests are closer to the nominal significance levels, whilst the remaining tests tend to be conservative. In summary, the five statistics based on the transformed sample, i.e. D, A^2, W^2, U^2 and V outperform the other tests in general, especially for small to medium-sized samples. Notably, Figure 3 shows that D^*, V^*, W^{2*}, U^{2*}, A^{2*} and r_n tests are still conservative for $n = 500$ when $r = 0$ and $b = 2.5$, corresponding to the symmetric distributions with flatter peaks and thinner tails compared to the normal one. The general conclusion is that the V and U^2-based tests are the most effective in maintaining the nominal significance level in most cases.

More conservative test results were observed for $r = 0, -0.2$ compared to $r = -0.7$. Figure 3 illustrates that for $r = -0.7$, a smaller sample size is required to maintain a 5% nominal level compared to $r = -0.2, 0$. Among the three values of r considered, the five standard tests based on the original data, i.e. D^*, V^*, W^{2*}, U^{2*}, A^{2*} appear to be the most conservative for the symmetric case ($r = 0$). However, it seems that the shape of the assumed distribution has little effect on the size. Globally, simulation results indicate that our proposed test procedure based on the approximately mixed gamma sample maintains the nominal significance level better than the standard tests based on the original data. Additionally, the r_n-based test tends to be conservative.

4.2. Power of the tests

In this subsection, we conduct a simulation study to investigate the power of the eleven GOF tests discussed in Section 3 under various alternative distributions. Specifically, we

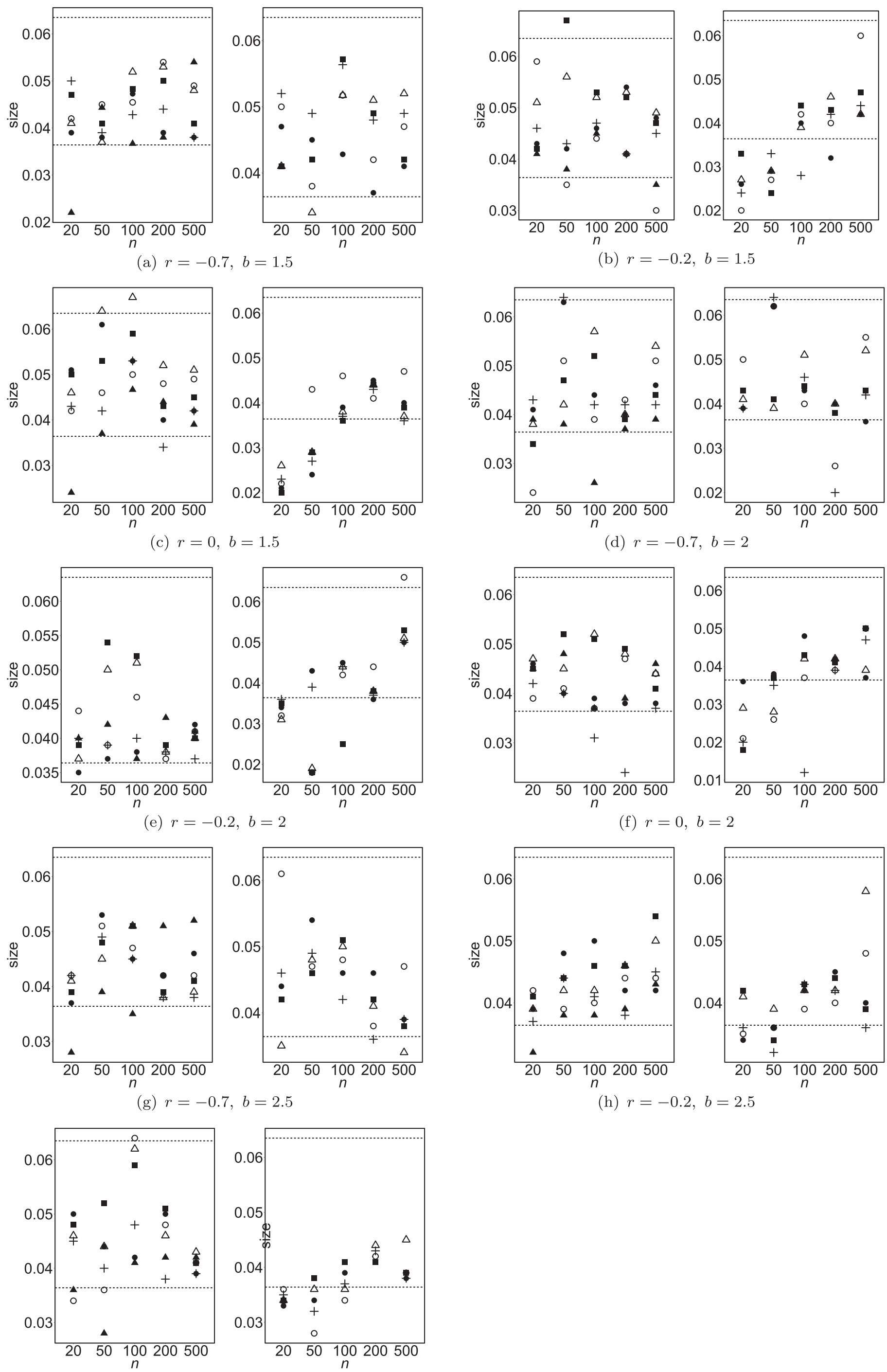


Figure 3. Empirical sizes of the GOF tests based on the $D(\circ)$, $V(\triangle)$, $W^2(+)$, $U^2(\blacksquare)$, $A^2(\bullet)$, $r_n(\blacktriangle)$, $D^*(\circ)$, $V^*(\triangle)$, $W^{2*}(+)$, $U^{2*}(\blacksquare)$ and $A^{2*}(\bullet)$ statistics when $\mathcal{F}$ is the SGN class and the nominal significance level is 5%. The left column of each subplot displays D, V, W^2, U^2, A^2 and r_n, while the right column displays D^*, V^*, W^{2*}, U^{2*} and A^{2*}. (a) $r = -0.7, b = 1.5$. (b) $r = -0.2, b = 1.5$. (c) $r = 0, b = 1.5$. (d) $r = -0.7, b = 2$. (e) $r = -0.2, b = 2$. (f) $r = 0, b = 2$. (g) $r = -0.7, b = 2.5$. (h) $r = -0.2, b = 2.5$ and (i) $r = 0, b = 2.5$.

analyse the rejection rates of the proposed tests against the 12 alternatives when the underlying distribution is assumed to be the SGN at a 5% significance level. These alternative distributions cover symmetric and asymmetric, unimodal and bimodal, leptokurtic and platykurtic, heavy-tailed and light-tailed densities. This allowed us to capture various possible data behaviours in practical situations. Simulations were carried out with a number of $N = 1000$ Monte Carlo samples of size n, using the ML method to estimate the null distribution.

We designed a simulation experiment to test the assumption of a SGN distribution when the data was in fact drawn from various alternative distributions. In this paper, we consider a general composite hypothesis given in (22) in ten cases. The simulated powers were estimated as follows:

(i) Generate $\mathbf{Z}_n$ from the alternative distribution;
(ii) Implement the tests given in Section 3 and record whether the test statistics reject the null hypothesis at the given significance level;
(iii) Repeat steps (i) and (ii) N times, and the power estimate was calculated as the percentage of rejected null hypotheses.

The simulated powers of the eleven tests are reported in Tables A1–A5.

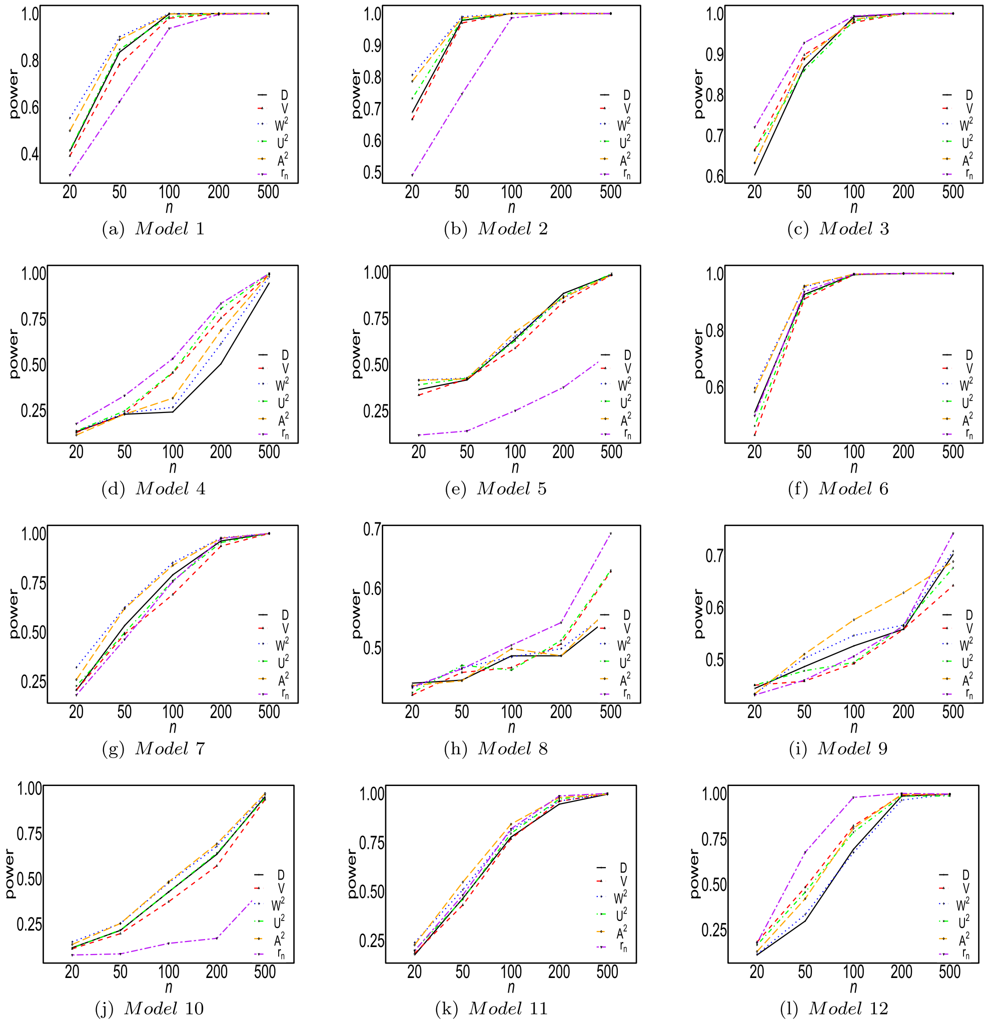
In the tables, the alternative distributions are labelled as follows: logistic ($\mathcal{L}$), Student (t_n) Cauchy ($\mathcal{C}$), Weibull ($\mathcal{W}$), Gumbel ($\mathcal{G}$), skew-normal ($\mathcal{SN}$), a mixture of normal ($Mix\,\mathcal{N}$), gamma (Ga) and chi-squared (χ^2), with the parameter value following in the parenthesis. For instance, $\chi^2(9)$ stands for the chi-squared distribution with the parameter equal to 9. Models 1–3 are symmetrical distributions with a flat peak, sharp peak, and extremely sharp peak, respectively. Models 4–6 are asymmetrical, leptokurtic, or platykurtic distributions, corresponding to heavy, canonical, and light tails, respectively. Model 6 is a four-parameter location-scale extension of the Azzalini-type $\mathcal{SN}$ distribution, with parameter vector $\boldsymbol{\theta}_1 = (\xi, \omega, \kappa, \tau)^\top$. Its density function at x is

$$f_{\mathcal{SN}}(x) = \frac{1}{\omega}\phi\left(\frac{x-\xi}{\omega}\right)\exp\left\{\Phi\left(\tau\sqrt{1+\kappa^2}+\kappa\frac{x-\xi}{\omega}\right)-\Phi(\tau)\right\}, \quad x \in R,$$

where κ is a shape parameter, and ϕ and Φ denote the standard normal density and distribution functions, respectively. The samples can be generated using the *sn* library of the $\mathbb{R}$ package. We set $\xi = 3, \omega = 2$ for the sake of clarity. Although $\mathcal{SN}$ distributions can be highly skewed, they can not exhibit bimodality like the mixture models. Thus, we have adopted the two-component normal mixture as alternative distributions in models 10-12. The data are generated from the following three models:

Model 10. $x \sim 0.7N(-1,2) + 0.3N(4,2)$,
Model 11. $x \sim 0.7N(2,2) + 0.3N(-1,1)$,
Model 12. $x \sim 0.7N(3,2) + 0.3N(-3,1)$.

Model 10 is a two-component normal mixture with equal variance, while Models 11 and 12 are two-component normal mixtures with unequal variance. Furthermore, we compare the tests against gamma alternatives with different parameters (see models 7–9). Notably, as the shape parameter ν_1 decreases, the gamma distribution becomes more skewed around



(a) *Model* 1 (b) *Model* 2 (c) *Model* 3 (d) *Model* 4 (e) *Model* 5 (f) *Model* 6 (g) *Model* 7 (h) *Model* 8 (i) *Model* 9 (j) *Model* 10 (k) *Model* 11 (l) *Model* 12

Figure 4. Empirical power, as a function of sample size, *n*, of parametric bootstrap goodness of fit tests when $\mathcal{F}$ is the SGN class. (a) *Model* 1. (b) *Model* 2. (c) *Model* 3. (d) *Model* 4. (e) *Model* 5. (f) *Model* 6. (g) *Model* 7. (h) *Model* 8. (i) *Model* 9. (j) *Model* 10. (k) *Model* 11 and (l) *Model* 12.

zero with lighter tails. Models 7–9 include three gamma distributions, corresponding to lighter, canonical, and heavier tails than logistic.

Firstly, we simulated n observations from each alternative model and estimated the unknown parameter under the SGN null hypothesis using the ML algorithm. Then, we calculated all test statistics: D, V, W^2, U^2, A^2, D^*, V^*, W^{2*}, U^{2*}, A^{2*} and r_n. The results are shown in Tables A1–A5. For each case, the best test is highlighted in boldface. It can be observed that, in general, tests based on the transformed sample perform better than the standard tests. Thus, power curves of D, V, W^2, U^2, A^2, and r_n tests as functions of n for different designs are plotted in Figure 4 for $n = 20, 50, 100, 200$ and 500. Figure 4 shows that the power of all the tests increases and approaches one with increasing sample size n, indicating that the tests based on the six statistics will be consistent.

Tables A1–A5 reveal the following findings. In the case of logistic and t_4 alternatives, W^2 is the most powerful test, while U^2 and A^2 have comparable powers, and the other tests are less powerful. In the case of Cauchy and Weibull alternatives, r_n appears to be the most powerful for all n, and D, V, W^2, U^2, A^2 have comparable powers. However, in the case of the Gumbel alternative, r_n performs worst overall, and the best-performing test is W^2-based test for small sample sizes $n = 20, 50$. For the $\mathcal{SN}$ alternative, the most powerful test is A^2, while D, V, W^2, U^2 have comparable powers. For both Gumbel and $\mathcal{SN}$ alternatives, none of the five tests based on the transformed sample consistently outperform any of the others for all n. In the case of gamma alternatives, r_n gives the best performance under large sample size n for $Ga(3, 0.8)$ alternative, while A^2 performs best in most cases. We compared the power performance for models 5, 6 (with infinite support) and models 4, 7, 8, 9 (with positive support) in Tables A3 and A4. The results suggest that, for skewed alternatives with infinite support, the powers are higher than those for alternatives with positive support, but this trend is observed only for sample sizes of 100 and 200. Additionally, for alternatives with infinite support or positive support, none of the tests consistently performed better.

An interesting conclusion is that, for $\chi^2(9)$ alternative, the five standard tests based on the original data, i.e. D^*, V^*, W^{2*}, U^{2*} and A^{2*} perform better for smaller n. However, as n increases, the proposed tests based on the transformed sample outperform the five standard tests. Similarly, for the bimodal alternative models, the powers of the five standard tests, D^*, V^*, W^{2*}, U^{2*}, and A^{2*}, are slightly higher than the five tests based on the transformation for models 10 and 12. However, as n increases, the difference in power between them decreases. The above results indicate that the ability of the proposed tests to discriminate whether small samples are drawn from a bimodal distribution is relatively limited, but this limitation can be mitigated by increasing sample size n. In addition, the r_n statistic emerges as the most powerful test for model 11 and model 12 under large sample sizes $n = 200$ and 500. In general, the W^2 and A^2-based tests perform the best in most cases.

Moreover, simulation results in tables also indicate that the tests are generally more powerful against symmetric alternatives than skewed models. This can be attributed to the different shapes of the SGN and symmetric distributions. Comparing models 4 and 9 with other asymmetric alternatives when $n \leq 50$, we observe that the powers of the tests are generally lowest against very heavy-tailed alternatives. In conclusion, for samples of small or moderate sizes from alternative distributions whose shapes are very different from those that SGN distributions can assume, the tests can be very powerful.

5. Data example: australian institute of sport data

As an illustrative example, we will analyse the Australian Institute of Sport (AIS) data, which consists of several biomedical measurements on 102 male and 100 female Australian athletes ([43]). The complete dataset is available as the *ais* data object in *R*'s *sn* package. Recently, [23] emphasized the significance of conducting goodness of fit tests in analysing the AIS data set using the sinh-arcsinh distributions. Additionally, [44,45] illustrated the performance of the proposed bivariate skew-normal distribution for some variables by using the AIS dataset. In this study, we focus on two variables: the weight (WT) (in kg) and body mass index (BMI) for male and female athletes. Table A6 summarizes the descriptive statistics. It can be seen that all variables exhibit varying degrees of skewness. In comparison to the WT variable, the BMI variable shows higher skewness and kurtosis. Table A7

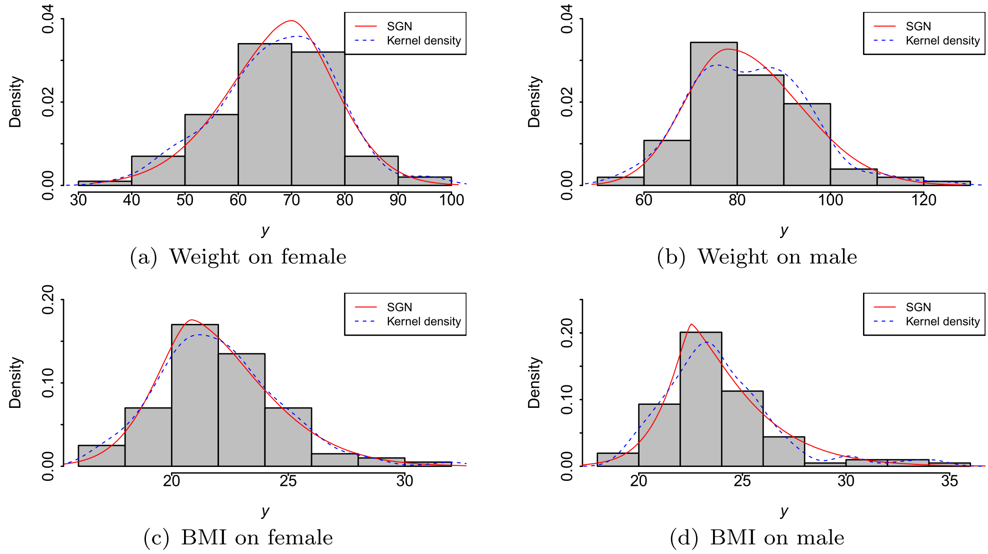


Figure 5. Histograms of the AIS dataset with an overlaid SGN fit and nonparametric kernel density estimation. (a) Weight on female; (b) Weight on male; (c) BMI on female; (d) BMI on male.

shows the p-values of all tests conducted on female and male athletes separately. These estimated p-values are estimated using the parametric bootstrap method with $B = 1000$ bootstrap samples. The lowest p-value for each variable is highlighted in bold. For the two variables of female and male athletes, all tests lead to the same conclusion in favor of the SGN distribution at any significance level below 12.3%.

To further investigate the distributional characteristics of the data, we conduct a detailed model analysis. The fitting results for the AIS dataset using the SGN distribution, obtained through the ML algorithm detailed in Section 2.1, are summarized in Table A7. The estimates of r in Table A7 support the presence of skewness in the underlying distribution of each variable. Figure 5 shows the histograms of the AIS dataset with an overlaid SGN fit and nonparametric kernel density estimation. It can be seen that all figures suggest that our SGN distribution offers a superior fit for each variable. Specifically, Figure 5 highlights the notable advantage of the SGN model in accurately capturing the sharp-peaked characteristics of male data for the BMI variable. Furthermore, the QQ plots in Figure 6 suggest that any lack-of-fit is most evident in the right-hand tail.

6. Discussion

In this paper, we develop two classes of goodness of fit tests for our SGN distribution based on data transformation, as well as the parametric bootstrap procedures. The test statistics are constructed based on the properties of the SGN family, and we investigate the main properties of the test statistics in detail. Simulation results demonstrate that the proposed test statistics effectively maintain the nominal significance level compared to the standard edf-based tests based on the original sample, especially for moderate to large sample sizes. The power study shows that the GOF tests based on the transformation are generally more powerful than the classical ones. Finally, a real example is given for an illustrative purpose.

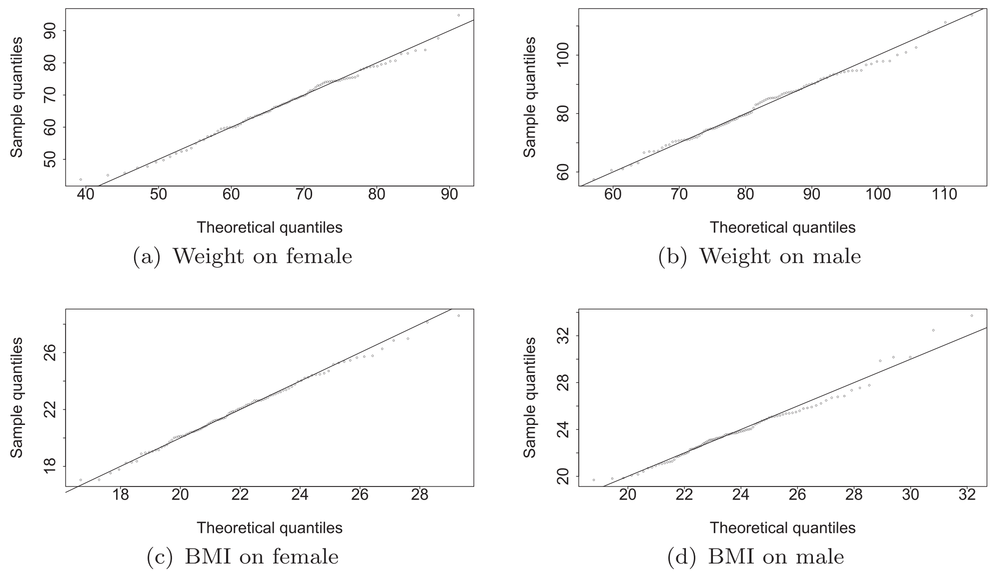


Figure 6. *QQ*—plots of two variables from the AIS data set for SGN fit: (a) Weight on female; (b) Weight on male; (c) BMI on female; (d) BMI on male.

An important issue for further research involves comparing the simulated performance of our SGN model with other published SGN models ([7,10]) when considering goodness of fit tests. However, it's important to note that the goodness of fit tests presented in our paper rely on unique properties of our SGN, such as the mixed gamma transformation. Directly applying these methods to existing SGN distributions may pose certain challenges. It would be interesting to explore more general methods for simultaneously evaluating the goodness of fit across various types of SGN distributions.

Acknowledgments

The authors sincerely thank the Associate Editor, and two reviewers for providing us with the opportunity to express our comments and suggestions. Their insightful feedback has been invaluable in refining and enhancing the quality of our manuscript. We would like to express our deepest gratitude to Professor Wei Ning in the Department of Mathematics and Statistics at Bowling Green State University, USA, and Professor Arthur Pewsey in the Mathematics Department at the University of Extremadura, Spain, for their helpful comments and assistance.

Disclosure statement

No potential conflict of interest was reported by the author(s).

References

[1] Eling M. Fitting insurance claims to skewed distributions: are the skew-normal and skew-student good models?. Insur Math Econ. 2012;51:239–248. doi: 10.1016/j.insmatheco.2012.04.001

[2] González-Estrada E, Cosmes W. Shapiro-wilk test for skew normal distributions based on data transformations. J Stat Comput Simul. 2019;89:3258–3272. doi: 10.1080/00949655.2019.1658763
[3] González-Estrada E, Villaseñor JA, Acosta-Pech R. Shapiro-wilk test for multivariate skew-normality. Comput Stat. 2022;37:1985–2001. doi: 10.1007/s00180-021-01188-y
[4] Amsler C, Papadopoulos A, Schmidt P. Evaluating the cdf of the skew normal distribution. Empir Econ. 2021;60:3171–3202. doi: 10.1007/s00181-020-01868-6
[5] Opperman L, Ning W. Sequential probability ratio test for skew normal distribution. Commun Stat Simul Comput. 2021;50:2823–2836. doi: 10.1080/03610918.2019.1614623
[6] Ye R, Fang B, Du W, et al. Bootstrap tests for the location parameter under the skew-normal population with unknown scale parameter and skewness parameter. Mathematics. 2022;10:921. doi: 10.3390/math10060921
[7] Rasekhi M, Hamedani G, Chinipardaz R. A flexible extension of skew generalized normal distribution. Metron. 2017;75:87–107. doi: 10.1007/s40300-017-0106-2
[8] Oskouei LH, Hasanalipour P, Khamnei HJ, et al. The beta skew-generalized normal distribution. Commun Stat Theory Methods. 2017;46:2270–2276. doi: 10.1080/03610926.2015.1041980
[9] Ferreira CS, Arellano-Valle RB. Estimation and diagnostic analysis in skew-generalized-normal regression models. J Stat Comput Simul. 2018;88:1039–1059. doi: 10.1080/00949655.2017.1419351
[10] Bekker A, Ferreira JT, Arashi M, et al. Computational methods applied to a skewed generalized normal family. Commun Stat Simul Comput. 2018;49:2930–2943. doi: 10.1080/03610918.2018.1530785
[11] Azzalini A. The skew-normal and related families. IMS Monographs. Cambridge: Cambridge University Press; 2014.
[12] Arellano-Valle RB, Gómez HW, Quintana FA. A new class of skew-normal distributions. Commun Stat Theory Methods. 2004;33:1465–1480. doi: 10.1081/STA-120037254
[13] Nekoukhou V, Alamatsaz M, Aghajani A. A flexible skew-generalized normal distribution. Commun Stat Theory Methods. 2013;42:2324–2334. doi: 10.1080/03610926.2011.599003
[14] Neethling A, Ferreira J, Bekker A, et al. Skew generalized normal innovations for the ar (p) process endorsing asymmetry. Symmetry. 2020;12:1253. doi: 10.3390/sym12081253
[15] Sulewski P, Szymkowiak M. Modelling income distributions based on theoretical distributions derived from normal distributions. Wiad Stat Polish Stat. 2023;68:1–23.
[16] Desgagné A, Lafaye de Micheaux P, Ouimet F. Goodness-of-fit tests for laplace, gaussian and exponential power distributions based on λ-th power skewness and kurtosis. Statistics. 2023;57:94–122. doi: 10.1080/02331888.2022.2144859
[17] Gupta AK, Chen T. Goodness-of-fit tests for the skew-normal distribution. Commun Stat Simul Comput. 2001;30:907–930. doi: 10.1081/SAC-100001854
[18] Mateu-Figueras G, Puig P, Pewsey A. Goodness-of-fit tests for the skew-normal distribution when the parameters are estimated from the data. Commun Stat Theory Methods. 2007;36:1735–1755. doi: 10.1080/03610920601126217
[19] Rodríguez PP, Alva JAV. On testing the skew normal hypothesis. J Stat Plan Inference. 2010;140:3148–3159. doi: 10.1016/j.jspi.2010.04.013
[20] Ning W, Ngunkeng G. An empirical likelihood ratio based goodness-of-fit test for skew normality. Stat Methods Appt. 2013;22:209–226. doi: 10.1007/s10260-012-0218-z
[21] Jiménez-Gamero MD, Kim HM. Fast goodness-of-fit tests based on the characteristic function. Comput Stat Data Anal. 2015;89:172–191. doi: 10.1016/j.csda.2015.03.015
[22] Balakrishnan N, Capitanio A, Scarpa B. A test for multivariate skew-normality based on its canonical form. J Multivar Anal. 2014;128:19–32. doi: 10.1016/j.jmva.2014.02.015
[23] Pewsey A. Parametric bootstrap edf-based goodness-of-fit testing for sinh–arcsinh distributions. Test. 2018;27:147–172. doi: 10.1007/s11749-017-0538-2
[24] Lian C, Rong Y, Cheng W. On a novel skewed generalized t distribution: properties, estimations and its applications. Publishing Taylor & Francis Online; 2024. [accessed 2024 February 21].

Availbale from: https://www.tandfonline.com/eprint/9HNTXT46S9THSQEYBEN3/full?target =10.1080/03610926.2024.2313034.
[25] Gómez HW, Salinas HS, Bolfarine H. Generalized skew-normal models: properties and inference. Statistics. 2006;40:495–505. doi: 10.1080/02331880600723168
[26] Azzalini A. Further results on a class of distributions which includes the normal ones. Statistica. 1986;46:199–208.
[27] Azzalini A. A class of distributions which includes the normal ones. Scand J Stat. 1985;12:171–178.
[28] Huber PJ. Robust statistics. Vol. 523. New Jersey: John Wiley & Sons; 2004.
[29] Nguyen TM, Wu QJ, Zhang H. Bounded generalized gaussian mixture model. Pattern Recognit. 2014;47:3132–3142. doi: 10.1016/j.patcog.2014.03.030
[30] Allili MS. Wavelet modeling using finite mixtures of generalized gaussian distributions: application to texture discrimination and retrieval. IEEE Trans Image Process. 2011;21:1452–1464. doi: 10.1109/TIP.2011.2170701
[31] Yalçınkaya A, Şenoğlu B, Yolcu U. Maximum likelihood estimation for the parameters of skew normal distribution using genetic algorithm. Swarm Evol Comput. 2018;38:127–138. doi: 10.1016/j.swevo.2017.07.007
[32] Alencar FHCD, Galarza CE, Matos LA, et al. Finite mixture modeling of censored and missing data using the multivariate skew-normal distribution. Adv Data Anal Classif. 2022;16:521–557. doi: 10.1007/s11634-021-00448-5
[33] Bickel DR. Robust estimators of the mode and skewness of continuous data. Comput Stat Data Anal. 2002;39:153–163. doi: 10.1016/S0167-9473(01)00057-3
[34] Lee S, McLachlan GJ. Finite mixtures of multivariate skew t-distributions: some recent and new results. Stat Comput. 2014;24:181–202. doi: 10.1007/s11222-012-9362-4
[35] Reiss RD. On the measurability and consistency of maximum likelihood estimates for unimodal densities. Ann Stat. 1973;1:888–901. doi: 10.1214/aos/1176342509
[36] Bryant P, Williamson JA. Asymptotic behaviour of classification maximum likelihood estimates. Biometrika. 1978;65:273–281. doi: 10.1093/biomet/65.2.273
[37] D'Agostino RB, Stephens MA. Goodness-of-fit techniques. NewYork: Marcel Dekker; 1986.
[38] Zamanzade E. Edf-based tests of exponentiality in pair ranked set sampling. Stat Pap. 2019;60:2141–2159. doi: 10.1007/s00362-017-0913-9
[39] Cabras S, Castellanos M. Default bayesian goodness-of-fit tests for the skew-normal model. J Appl Stat. 2009;36:223–232. doi: 10.1080/02664760802443988
[40] Lawless JF. Statistical models and methods for lifetime data. New York: John Wiley & Sons; 2011.
[41] Thadewald T, Büning H. Jarque–bera test and its competitors for testing normality–a power comparison. J Appl Stat. 2007;34:87–105. doi: 10.1080/02664760600994539
[42] Pewsey A, Kato S. Parametric bootstrap goodness-of-fit testing for wehrly–johnson bivariate circular distributions. Stat Comput. 2016;26:1307–1317. doi: 10.1007/s11222-015-9605-2
[43] Cook RD, Weisberg S. An introduction to regression graphics. New York: John Wiley & Sons; 1994.
[44] Azzalini A, Valle AD. The multivariate skew-normal distribution. Biometrika. 1996;83:715–726. doi: 10.1093/biomet/83.4.715
[45] Azzalini A, Capitanio A. Statistical applications of the multivariate skew normal distribution. J R Stat Soc Ser B (Stat Methodol). 1999;61:579–602. doi: 10.1111/1467-9868.00194

Appendices

Appendix 1. Some proofs

Appendix 1 provides some proofs referred to in Section 2.

Table A1. Empirical powers of the GOF tests based on D, V, W^2, U^2, A^2, D^*, V^*, W^{2*}, U^{2*}, A^{2*} and r_n statistics when $\mathcal{F}$ is the SGN class; Significance level $\alpha = 5\%$; Sample size $n = 20$.

		Transformation					Original					
Alternative		D	V	W^2	U^2	A^2	D^*	V^*	W^{2*}	U^{2*}	A^{2*}	r_n
Sym. dist.												
1	$\mathcal{L}(1,2)$	0.410	0.390	**0.552**	0.414	0.498	0.368	0.338	0.448	0.360	0.416	0.308
2	t_4	0.688	0.666	0.806	0.732	0.786	0.786	0.708	**0.836**	0.790	0.820	0.490
3	$\mathcal{C}(1,0.2)$	0.602	0.664	0.632	0.662	0.632	0.516	0.598	0.568	0.610	0.532	**0.720**
Asym. dist.												
4	$\mathcal{W}(2)$	0.132	0.128	0.136	0.136	0.114	0.140	0.108	0.136	0.116	0.114	**0.176**
5	$\mathcal{G}(0.2,1.4)$	0.364	0.334	**0.416**	0.390	0.414	0.350	0.326	0.410	0.380	0.410	0.118
6	$\mathcal{SN}(4,-0.7)$	0.510	0.430	**0.596**	0.462	0.582	0.250	0.244	0.348	0.268	0.312	0.498
7	$Ga(3,0.8)$	**0.440**	0.420	0.432	0.424	0.436	0.432	0.418	0.428	0.422	0.428	0.434
8	$Ga(2.2,0.5)$	0.444	**0.450**	0.434	**0.450**	0.434	0.415	0.446	0.413	0.444	0.420	0.432
9	$\chi^2(9)$	0.118	0.112	0.150	0.116	0.136	0.142	0.126	**0.174**	0.126	0.156	0.078
Bimo. dist.												
10	$Mix\,\mathcal{N}_1$	0.204	0.206	0.320	0.226	0.258	0.304	0.200	**0.352**	0.238	0.322	0.180
11	$Mix\,\mathcal{N}_2$	0.175	0.183	**0.238**	0.199	0.226	0.217	0.188	0.227	0.206	0.227	0.196
12	$Mix\,\mathcal{N}_3$	0.110	**0.181**	0.113	0.165	0.129	0.116	0.168	0.091	0.134	0.109	0.175

Table A2. Empirical powers of the GOF tests based on D, V, W^2, U^2, A^2, D^*, V^*, W^{2*}, U^{2*}, A^{2*} and r_n statistics when $\mathcal{F}$ is the SGN class; Significance level $\alpha = 5\%$; Sample size $n = 50$.

		Transformation					Original					
Alternative		D	V	W^2	U^2	A^2	D^*	V^*	W^{2*}	U^{2*}	A^{2*}	r_n
Sym. dist.												
1	$\mathcal{L}(1,2)$	**0.585**	0.582	0.481	0.482	0.442	0.472	0.404	0.434	0.411	0.418	0.452
2	t_4	0.978	0.970	**0.990**	0.980	0.986	0.976	0.978	0.988	0.984	0.982	0.746
3	$\mathcal{C}(1,0.2)$	0.868	0.898	0.888	0.860	0.888	0.864	0.874	0.866	0.852	0.880	**0.928**
Asym. dist.												
4	$\mathcal{W}(2)$	0.228	0.230	0.234	0.246	0.228	0.216	0.226	0.220	0.242	0.220	**0.330**
5	$\mathcal{G}(0.2,1.4)$	0.416	0.422	**0.426**	**0.426**	0.422	0.416	0.422	0.424	0.424	0.424	0.140
6	$\mathcal{SN}(4,-0.7)$	0.926	0.910	0.952	0.920	**0.956**	0.614	0.566	0.712	0.596	0.694	0.936
7	$Ga(3,0.8)$	0.445	0.458	0.466	**0.470**	0.444	0.446	0.438	0.458	0.452	0.430	0.464
8	$Ga(2.2,0.5)$	0.486	0.458	0.502	0.478	**0.510**	0.484	0.456	0.474	0.480	0.460	0.460
9	$\chi^2(9)$	0.214	0.196	**0.252**	0.216	**0.252**	0.182	0.158	0.220	0.200	0.210	0.084
Bimo. dist.												
10	$Mix\,\mathcal{N}_1$	0.532	0.488	0.624	0.496	0.618	0.626	0.562	**0.706**	0.612	0.702	0.462
11	$Mix\,\mathcal{N}_2$	0.464	0.430	0.508	0.459	**0.547**	0.422	0.367	0.511	0.395	0.482	0.481
12	$Mix\,\mathcal{N}_3$	0.297	0.483	0.333	0.456	0.420	0.461	0.543	0.331	0.493	0.502	**0.676**

A.1 Proof of Proposition 2.2

Proof: As $r \neq 0, \pm 1$, it follows from Proposition 2.1 that

$$\frac{|X-\mu|^b}{\sigma^b} \overset{d}{=} 2 \cdot |W|^b Y,$$

where $2Y \sim Ga(1/b, 1/2)$. Then we have

$$T \overset{d}{=} \frac{1-r}{2} \cdot Ga\left(\frac{1}{b}, \lambda_1\right) + \frac{1+r}{2} \cdot Ga\left(\frac{1}{b}, \lambda_2\right). \tag{A1}$$

Similarly, the distribution of T can be obtained for $r = 0$ and $r \neq 0, \pm 1$.

■

We establish consistency of the ML estimators of all parameters of the SGN distribution.

Table A3. Empirical powers of the GOF tests based on $D, V, W^2, U^2, A^2, D^*, V^*, W^{2*}, U^{2*}, A^{2*}$ and r_n statistics when $\mathcal{F}$ is the SGN class; Significance level $\alpha = 5\%$; Sample size $n = 100$.

		Transformation					Original					
Alternative		D	V	W^2	U^2	A^2	D^*	V^*	W^{2*}	U^{2*}	A^{2*}	r_n
1	$\mathcal{L}(1,2)$	0.784	0.788	0.781	0.782	**0.802**	0.672	0.704	0.731	0.661	0.708	0.712
2	t_4	0.980	0.972	**0.992**	0.983	0.988	0.979	0.98	0.922	0.986	0.985	0.821
3	$\mathcal{C}(1,0.2)$	0.888	0.912	0.911	0.892	0.926	0.884	0.895	0.892	0.874	0.912	**0.928**
Asym. dist.												
4	$\mathcal{W}(2)$	0.240	0.454	0.266	0.458	0.316	0.240	0.414	0.258	0.432	0.298	**0.532**
5	$\mathcal{G}(0.2,1.4)$	0.638	0.588	0.652	0.628	**0.676**	0.636	0.572	0.638	0.632	0.660	0.250
6	$\mathcal{SN}(4,-0.7)$	**0.921**	0.915	0.826	0.831	0.862	0.721	0.651	0.810	0.644	0.781	0.624
7	$Ga(3,0.8)$	0.486	0.466	0.484	0.462	0.498	0.480	0.434	0.472	0.452	0.494	**0.504**
8	$Ga(2.2,0.5)$	0.526	0.492	0.546	0.494	**0.576**	0.516	0.490	0.504	0.492	0.480	0.506
9	$\chi^2(9)$	0.426	0.372	0.476	0.426	**0.482**	0.402	0.356	0.454	0.428	0.452	0.142
Bimo. dist.												
10	$Mix\,\mathcal{N}_1$	0.792	0.791	0.831	0.731	**0.853**	0.776	0.758	0.786	0.658	0.783	0.721
11	$Mix\,\mathcal{N}_2$	0.777	0.768	0.812	0.796	**0.843**	0.677	0.617	0.774	0.619	0.755	0.819
12	$Mix\,\mathcal{N}_3$	0.692	0.821	0.674	0.786	0.808	0.850	0.891	0.777	0.873	0.923	**0.978**

Table A4. Empirical powers of the GOF tests based on $D, V, W^2, U^2, A^2, D^*, V^*, W^{2*}, U^{2*}, A^{2*}$ and r_n statistics when $\mathcal{F}$ is the SGN class; Significance level $\alpha = 5\%$; Sample size $n = 200$.

		Transformation					Original					
Alternative		D	V	W^2	U^2	A^2	D^*	V^*	W^{2*}	U^{2*}	A^{2*}	r_n
1	$\mathcal{L}(1,2)$	0.884	0.888	0.884	0.882	**0.891**	0.782	0.784	0.811	0.861	0.858	0.812
2	t_4	0.984	0.978	**0.994**	0.986	0.991	0.982	0.984	0.941	0.991	0.989	0.912
3	$\mathcal{C}(1,0.2)$	0.913	0.932	0.932	0.924	**0.946**	0.916	0.913	0.914	0.892	0.936	0.943
Asym. dist.												
4	$\mathcal{W}(2)$	0.504	0.754	0.614	0.806	0.686	0.486	0.672	0.558	0.716	0.606	**0.834**
5	$\mathcal{G}(0.2,1.4)$	**0.884**	0.838	0.860	0.870	0.862	0.878	0.838	0.854	0.864	0.850	0.376
6	$\mathcal{SN}(4,-0.7)$	**0.982**	0.973	0.981	0.976	0.962	0.804	0.778	0.880	0.784	0.874	0.798
7	$Ga(3,0.8)$	0.486	0.506	0.498	0.512	0.486	0.480	0.476	0.494	0.466	0.492	**0.542**
8	$Ga(2.2,0.5)$	0.558	0.558	0.566	0.566	**0.628**	0.526	0.494	0.598	0.526	0.512	0.564
9	$\chi^2(9)$	0.634	0.570	0.678	0.640	**0.692**	0.600	0.556	0.662	0.644	0.678	0.170
Bimo. dist.												
10	$Mix\,\mathcal{N}_1$	0.892	0.894	0.911	0.894	**0.916**	0.876	0.880	0.886	0.858	0.883	0.891
11	$Mix\,\mathcal{N}_2$	0.945	0.960	0.961	0.971	0.979	0.917	0.888	0.959	0.894	0.951	**0.987**
12	$Mix\,\mathcal{N}_3$	0.985	0.991	0.963	0.985	0.995	0.998	0.998	0.993	0.998	**1.000**	**1.000**

A.2 Proof of Theorem 2.1

To investigate the consistency of the ML estimators for the SGN distribution, the following Lemma given in [35] is needed.

Lemma A.1 ([35]): *Let $\mathscr{H}^a$ be the family of all unimodal probability densities with mode at $a \in \mathbb{R}$. Any sequence of ML estimates for $\mathscr{H}^a$ is strongly consistent for each $f \in \mathscr{H}^a$ fulfilling*

(i) $\int f(x) \ln|x-a|\,\mathrm{d}x > -\infty$ *and,*
(ii) $\int f(x) \ln f(x)\,\mathrm{d}x > -\infty$.

The support of $f_{SGN}(x;\boldsymbol{\theta})$ is independent of $\boldsymbol{\theta}$ for any $\boldsymbol{\theta} \in \Theta$. Thus, the consistency of the ML estimate $\widehat{\boldsymbol{\theta}}_{MLE}$ for the SGN distribution can be shown by verifying the conditions in Lemma A.1.

Table A5. Empirical powers of the GOF tests based on $D, V, W^2, U^2, A^2, D^*, V^*, W^{2*}, U^{2*}, A^{2*}$ and r_n statistics when $\mathcal{F}$ is the SGN class; Significance level $\alpha = 5\%$; Sample size $n = 500$.

		Transformation					Original					
Alternative		D	V	W^2	U^2	A^2	D^*	V^*	W^{2*}	U^{2*}	A^{2*}	r_n
Sym. dist.												
1	$\mathcal{L}(1,2)$	0.984	0.978	**0.994**	0.986	0.991	0.982	0.984	0.941	0.991	0.989	0.912
2	t_4	0.988	0.981	**0.997**	0.989	0.993	0.985	0.989	0.961	0.996	0.993	0.942
3	$\mathcal{C}(1,0.2)$	0.943	0.955	0.952	0.941	0.963	0.947	0.942	0.952	0.932	0.965	**0.973**
Asym. dist.												
4	$\mathcal{W}(2)$	0.946	0.994	0.980	**0.998**	0.992	0.934	0.964	0.950	0.980	0.954	**0.998**
5	$\mathcal{G}(0.2,1.4)$	0.986	0.984	0.984	**0.992**	0.984	0.984	0.982	0.978	**0.992**	0.976	0.566
6	$\mathcal{SN}(4,-0.7)$	0.996	0.996	**0.998**	0.996	**0.998**	0.914	0.982	0.935	0.934	0.931	0.985
7	$Ga(3,0.8)$	0.552	0.628	0.562	0.630	0.568	0.514	0.484	0.542	0.484	0.528	**0.692**
8	$Ga(2.2,0.5)$	0.702	0.642	0.708	0.676	0.688	0.700	0.552	0.694	0.588	0.676	**0.742**
9	$\chi^2(9)$	0.952	0.936	0.962	0.944	**0.970**	0.946	0.936	0.958	0.944	0.966	0.454
Bimo. dist.												
10	Mix $\mathcal{N}_1$	0.986	0.980	**0.996**	0.988	**0.996**	0.962	0.936	0.978	0.954	0.976	0.974
11	Mix $\mathcal{N}_2$	0.996	0.998	0.996	**1.000**	0.996	**1.000**	0.998	**1.000**	0.996	**1.000**	**1.000**
12	Mix $\mathcal{N}_3$	0.996	0.994	0.996	0.986	0.996	0.958	0.984	0.956	0.986	0.959	**0.997**

Specifically, we need to check the following two equations:

$$E_{\boldsymbol{\theta}}\left[\ln f_{SGN}(X;\boldsymbol{\theta})\right] > -\infty \tag{A2}$$

and

$$E_{\boldsymbol{\theta}}\left[\ln|X-\mu|\right] > -\infty, \tag{A3}$$

for any $\boldsymbol{\theta} \in \Theta$.

Let $X \sim \mathrm{SGN}(\mu,\sigma,r,b)$, we have $G = \frac{|X-\mu|^b}{2\sigma^b[1+r\cdot\mathrm{sign}(x-\mu)]^b)} \sim Ga(1/b,1)$ from Equation (3). Thus, condition (A1) is easily verified for any $\boldsymbol{\theta} \in \Theta$.

Next, we check condition (A2). Note that

$$\begin{aligned}
&\int f_{SGN}(x;\mu,\sigma,r,b)\ln|x-\mu|\,\mathrm{d}x \\
&= \frac{b}{2^{1+1/b}\Gamma(1/b)\sigma}\left\{\int_0^\infty \ln(y)\exp\left\{\frac{y^b}{-2\sigma^b(1+r)^b}\right\}\mathrm{d}y\right. \\
&\quad \left. + \int_0^\infty \ln(y)\exp\left\{\frac{y^b}{-2\sigma^b(1-r)^b}\right\}\mathrm{d}y\right\}.
\end{aligned}$$

Let $z = \frac{y}{\sigma(1+r)}$ for the first integral, then we only need to prove

$$\int_0^\infty \ln(z)\exp\left\{-\frac{z^b}{2}\right\}\mathrm{d}z > -\infty.$$

For any $\boldsymbol{\theta} \in \Theta$,

$$\int_0^\infty \ln(z)\exp\left\{-\frac{z^b}{2}\right\}\mathrm{d}z = \frac{2^{1/b}}{b^2}\Gamma(1/b)[\ln(2)+\psi(1/b)] > -\infty$$

can be easily justify, where $\psi(\cdot)$ denotes the digamma function. Thus condition (A3) is satisfied for the SGN distribution.

A.3 Proof of Theorem 2.2

We prove this theorem by computing the expectations in Equation (20). First, we present several results necessary for computing the elements of the information matrix

$$\mathrm{E}\left[\frac{|X-\mu|^k}{\sigma^k}\mathbb{I}_{[\mu,+\infty)}(X)\right]=\frac{2^{\frac{k}{b}-1}(1+r)^{k+1}}{\Gamma(1/b)}\Gamma\left(\frac{k+1}{b}\right),$$

$$\mathrm{E}\left[\frac{|X-\mu|^k}{\sigma^k}\mathbb{I}_{(-\infty,\mu)}(X)\right]=\frac{2^{\frac{k}{b}-1}(1-r)^{k+1}}{\Gamma\left(\frac{1}{b}\right)}\Gamma\left(\frac{k+1}{b}\right),$$

$$\mathrm{E}\left[\frac{|X-\mu|^k}{\sigma^k}\ln\left(\frac{|X-\mu|}{\sigma\left[1+r\cdot\operatorname{sign}(X-\mu)\right]}\right)\mathbb{I}_{[\mu,+\infty)}(X)\right]$$
$$=\frac{2^{\frac{k}{b}-1}(1+r)^{k+1}\left[\Gamma\left(\frac{k+1}{b}\right)\ln 2+\Gamma'\left(\frac{k+1}{b}\right)\right]}{\Gamma\left(\frac{1}{b}\right)b},$$

$$\mathrm{E}\left[\frac{|X-\mu|^k}{\sigma^k}\ln\left(\frac{|X-\mu|}{\sigma\left[1+r\cdot\operatorname{sign}(X-\mu)\right]}\right)\mathbb{I}_{(-\infty,\mu)}(X)\right]$$
$$=\frac{2^{\frac{k}{b}-1}(1-r)^{k+1}\left[\Gamma\left(\frac{k+1}{b}\right)\ln 2+\Gamma'\left(\frac{k+1}{b}\right)\right]}{\Gamma\left(\frac{1}{b}\right)b},$$

$$\mathrm{E}\left[\frac{|X-\mu|^k}{\sigma^k}\left(\ln\left(\frac{|X-\mu|}{\sigma\left[1+r\cdot\operatorname{sign}(X-\mu)\right]}\right)\right)^2\mathbb{I}_{[\mu,+\infty)}(X)\right]$$
$$=\frac{2^{\frac{k}{b}-1}(1+r)^{k+1}\left[\Gamma\left(\frac{k+1}{b}\right)(\ln 2)^2+2\Gamma'\left(\frac{k+1}{b}\right)\ln 2+\Gamma''\left(\frac{k+1}{b}\right)\right]}{\Gamma\left(\frac{1}{b}\right)b^2},$$

$$\mathrm{E}\left[\frac{|X-\mu|^k}{\sigma^k}\left(\ln\left(\frac{|X-\mu|}{\sigma\left[1+r\cdot\operatorname{sign}(X-\mu)\right]}\right)\right)^2\mathbb{I}_{(-\infty,\mu)}(X)\right]$$
$$=\frac{2^{\frac{k}{b}-1}(1-r)^{k+1}\left[\Gamma\left(\frac{k+1}{b}\right)(\ln 2)^2+2\Gamma'\left(\frac{k+1}{b}\right)\ln 2+\Gamma''\left(\frac{k+1}{b}\right)\right]}{\Gamma\left(\frac{1}{b}\right)b^2},$$

where $\mathbb{I}_{\mathbb{A}}(x)$ denotes the indicator function of the set $\mathbb{A}$. Next, by computing the second-order derivatives of the complete log-likelihood function in Equation (8) with respect to all parameters and taking expectations, we arrive at the following components:

$$I_{11}=\frac{nb(b-1)\Gamma(1-1/b)}{2^{2/b}\left(1-r^2\right)\sigma^2\Gamma(1/b)},$$

$$I_{13}=\frac{nb^2}{2^{1/b}\left(1-r^2\right)\sigma\Gamma(1/b)},$$

$$I_{22}=\frac{nb}{\sigma^2},$$

$$I_{24}=-\frac{n}{\sigma b}\left[\ln 2+1+\psi(1+1/b)\right],$$

Table A6. Descriptive statistics for AIS dataset.

	WT		BMI	
Statistic	Female	male	Female	male
Mean	67.3425	82.5235	21.9892	23.9036
SD	10.9154	12.4062	2.6400	2.7675
Skewness	−0.1697	0.3844	0.6830	1.3906
Kurtosis	0.07110	0.3430	1.0930	2.8694

$$I_{33} = \frac{n(b+1)}{1-r^2},$$

$$I_{44} = \frac{n}{b^2} + \frac{2n}{b^3}\ln 2 + \frac{2n}{b^3}\psi\left(\frac{1}{b}\right) + \frac{n}{b^4}\psi'\left(\frac{1}{b}\right)$$
$$+ \frac{n}{b^3}\left[(\ln 2)^2 + 2\psi(1+1/b)\ln 2 + \psi'\left(1+\frac{1}{b}\right) + \psi^2\left(1+\frac{1}{b}\right)\right],$$

$$I_{12} = I_{14} = I_{23} = I_{34} = 0,$$

where $\psi(\cdot)$ and $\psi'(\cdot)$ are defined in Equation (17).

Appendix 2. Additional simulations

A.4 Complementary results of Section 4.1

Appendix 2 provides the empirical sizes of the 11 tests in Section 3 with significance levels of $\alpha = 0.1$ and 0.01.

A.5 Sensitivity analysis

To explore the sensitivity of the tests to parameter estimates, we conducted a simulation study to provide empirical evidence. We use the same simulation settings as described in Section 4.1, with the only difference being the direct substitution of the true SGN model parameters into the test statistics after generating the sample, instead of using ML estimates. Subsequently, we calculated the empirical sizes of eleven tests across different sample sizes and distributions of data. The results, obtained under the nominal significance level of 5%, are summarized in Figure A1.

Comparing the results in Figures 3 and A1, we observe no significant variation in the empirical sizes between the tests based on the true parameter values and the tests based on ML estimates. This suggests that our tests are not overly sensitive to variations in parameter estimates. In other words, the performances of tests remain stable within the allowable variance range of the ML estimates. For other nominal significance levels, similar conclusions are obtained, so they are not shown to save space but can be provided upon request of the authors.

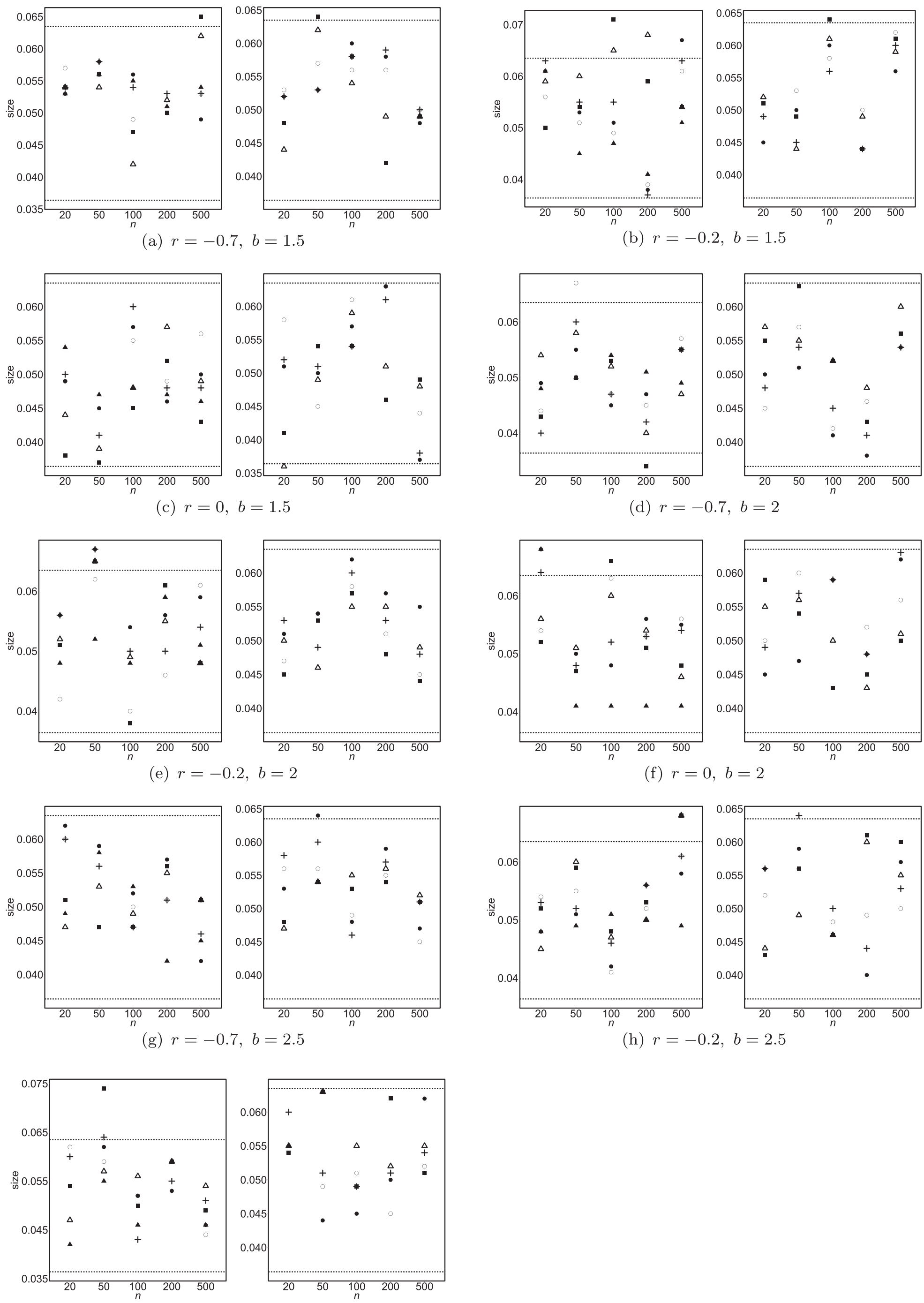


Figure A1. When true parameter values replace ML estimates in test statistics: Empirical sizes of the GOF tests based on the $D(\circ)$, $V(\triangle)$, $W^2(+)$, $U^2(\blacksquare)$, $A^2(\bullet)$, $r_n(\blacktriangle)$, $D^*(\circ)$, $V^*(\triangle)$, $W^{2*}(+)$, $U^{2*}(\blacksquare)$ and $A^{2*}(\bullet)$ statistics when $\mathcal{F}$ is the SGN class and the nominal significance level is 5%. The left column of each subplot displays D, V, W^2, U^2, A^2 and r_n, while the right column displays D^*, V^*, W^{2*}, U^{2*} and A^{2*}. (a) $r = -0.7, b = 1.5$. (b) $r = -0.2, b = 1.5$. (c) $r = 0, b = 1.5$. (d) $r = -0.7, b = 2$. (e) $r = -0.2, b = 2$. (f) $r = 0, b = 2$. (g) $r = -0.7, b = 2.5$. (h) $r = -0.2, b = 2.5$ and (i) $r = 0, b = 2.5$.

Table A7. AIS data set for the SGN fit: Estimated p-values of parametric bootstrap goodness of fit tests based on the proposed statistics calculated using $B = 1000$ bootstrap samples, along with parameter estimates and standard errors (obtained through an information-based method) in parenthesis.

Variables	WT		BMI	
Sex	Female	male	Female	male
Test statistics				
D	0.610	0.299	0.995	0.255
V	0.598	0.319	0.981	0.156
W^2	0.628	0.314	0.968	0.154
U^2	0.571	**0.286**	0.964	0.131
A^2	0.674	0.429	0.971	0.134
D^*	0.517	0.654	0.993	0.446
V^*	**0.408**	0.416	0.986	0.403
W^{2*}	0.819	0.509	0.992	0.237
U^{2*}	0.760	0.387	0.99	0.261
A^{2*}	0.770	0.529	0.987	0.127
r_n	0.713	0.350	**0.679**	**0.123**
Estimates				
μ	70.0073(2.6598)	77.9383(3.1053)	20.8365(0.5367)	22.5201(0.1251)
σ	9.5802(2.1475)	12.1965(2.2285)	2.0729(0.5065)	1.4715(0.4368)
r	−0.1552(0.1479)	0.2332(0.1475)	0.2900(0.1313)	0.3708(0.0686)
b	1.7620 (0.3660)	1.9972 (0.3890)	1.6243 (0.333)	1.2824 (0.2376)
$\ell(\widehat{\boldsymbol{\theta}})$	−379.7516	−399.9719	−235.3707	−236.9480

Notes: The log-likelihood (denoted as $\ell(\widehat{\boldsymbol{\theta}})$) values are also presented for the four considered variables. Bold font is used to highlight the lowest estimated p-values among the eleven test statistics.

Table A8. Empirical size of goodness of fit tests based on the $D, V, W^2, U^2, A^2, D^*, V^*, W^{2*}, U^{2*}, A^{2*}$ and r_n statistics when $\mathcal{F}$ is the SGN class and the nominal significance level is 1%.

r&*b*		$r=-0.7$					$r=-0.2$					$r=0$				
	n	20	50	100	200	500	20	50	100	200	500	20	50	100	200	500
$b=1.5$	D	0.014	0.014	0.005	0.008	0.006	0.016	0.004	0.006	0.008	0.014	0.012	0.006	0.008	0.006	0.012
	V	0.014	0.012	0.005	0.016	0.008	0.008	0.004	0.012	0.010	0.012	0.012	0.010	0.010	0.014	0.014
	W^2	0.012	0.010	**0.004**	0.012	0.012	0.012	**0.002**	0.010	**0.002**	0.016	0.016	0.004	0.010	0.012	0.010
	U^2	0.016	0.016	0.007	0.014	0.008	0.012	**0.002**	*0.018*	0.010	0.016	0.014	0.016	0.012	0.010	0.016
	A^2	0.012	0.010	0.013	0.010	0.006	0.012	**0.000**	0.010	0.012	0.016	0.016	0.014	0.012	0.016	0.012
	D^*	0.016	0.004	0.013	0.012	0.008	0.004	0.004	0.006	**0.002**	0.008	0.004	0.008	0.006	0.010	0.008
	V^*	0.014	0.008	*0.020*	0.014	0.008	0.004	0.008	0.010	**0.000**	0.014	0.004	0.008	0.008	0.004	0.006
	W^{2*}	0.012	0.012	0.009	0.008	0.012	0.008	0.004	0.006	**0.002**	0.004	0.004	0.004	0.006	**0.002**	0.006
	U^{2*}	0.016	0.008	*0.020*	0.012	0.016	0.006	0.008	0.008	**0.002**	0.006	0.004	0.004	0.006	**0.002**	0.010
	A^{2*}	0.012	*0.020*	0.011	0.008	0.010	0.008	0.008	0.010	0.004	0.008	0.004	0.008	0.006	**0.002**	0.010
	r_n	**0.002**	**0.003**	0.010	0.014	0.006	**0.002**	0.007	**0.000**	0.006	**0.001**	**0.000**	**0.000**	0.007	0.004	0.010
$b=2$	D	0.006	*0.022*	*0.024*	0.006	0.008	0.012	0.008	0.008	0.004	0.006	0.006	0.006	0.006	0.010	0.010
	V	0.006	0.008	0.012	0.008	0.016	0.008	0.014	*0.018*	0.004	0.004	0.012	0.014	0.008	0.006	0.014
	W^2	0.010	*0.040*	*0.024*	**0.002**	0.014	0.004	0.006	0.006	0.004	0.004	0.010	0.010	**0.002**	0.006	0.012
	U^2	0.006	0.016	*0.018*	0.004	0.016	0.010	0.010	0.012	0.006	0.004	0.010	0.014	0.006	0.004	0.014
	A^2	0.010	*0.036*	*0.024*	**0.002**	0.016	0.008	0.006	0.004	**0.002**	0.006	0.008	0.004	0.006	0.008	0.014
	D^*	0.010	*0.028*	*0.024*	0.008	0.014	0.008	0.006	**0.000**	**0.002**	0.006	**0.002**	0.006	0.008	**0.002**	0.006
	V^{2*}	0.004	0.008	*0.018*	0.008	0.016	0.012	0.008	**0.002**	0.008	0.010	0.004	**0.002**	0.004	0.004	0.008
	W^{2*}	*0.018*	*0.042*	*0.024*	0.004	0.014	0.008	**0.000**	**0.000**	0.006	0.010	**0.000**	**0.002**	**0.002**	0.004	0.014
	U^{2*}	0.006	0.016	*0.022*	0.004	0.016	0.008	0.004	**0.000**	0.008	0.010	**0.000**	0.004	**0.002**	0.004	0.014
	A^{2*}	0.012	*0.044*	*0.024*	0.004	*0.018*	0.006	**0.000**	**0.000**	0.006	0.008	**0.000**	**0.002**	**0.002**	**0.002**	*0.018*
	r_n	0.006	0.008	0.006	**0.002**	0.006	**0.002**	**0.000**	**0.002**	0.008	0.008	**0.000**	**0.002**	**0.002**	0.004	0.010
$b=2.5$	D	0.014	0.010	*0.020*	0.004	*0.018*	0.010	0.012	0.010	0.008	0.004	0.006	0.010	0.014	0.004	0.004
	V	0.010	0.010	*0.020*	0.010	0.016	0.010	0.014	0.014	0.008	0.014	0.006	0.004	*0.018*	0.006	0.004
	W^2	0.012	*0.018*	*0.020*	0.012	0.010	0.012	0.008	0.012	0.004	0.006	0.008	0.008	0.012	0.008	0.004
	U^2	0.008	0.014	0.016	0.014	0.010	0.008	0.010	0.010	0.008	0.006	0.010	0.008	*0.018*	0.012	0.006
	A^2	0.014	*0.018*	*0.026*	0.010	0.012	0.010	0.012	0.016	0.006	**0.002**	0.006	0.012	*0.018*	0.012	0.006
	D^*	0.016	0.016	*0.022*	0.004	0.014	0.010	**0.000**	0.010	0.008	0.010	0.004	0.008	0.006	0.004	**0.002**
	V^*	0.010	0.008	0.016	0.008	0.014	0.010	0.004	0.010	0.008	0.006	0.006	0.006	0.014	0.010	0.006
	W^{2*}	0.012	*0.028*	*0.022*	0.006	0.006	0.008	**0.000**	0.008	0.008	0.006	**0.002**	0.010	**0.002**	0.004	**0.002**
	U^{2*}	0.008	0.014	0.016	0.012	0.008	0.008	**0.000**	0.012	0.008	0.010	0.006	0.004	0.010	0.006	0.004
	A^{2*}	0.012	*0.028*	*0.022*	0.004	0.006	0.008	**0.000**	0.008	0.010	**0.002**	**0.002**	0.010	0.004	0.004	0.004
	r_n	0.012	0.014	0.008	0.010	0.010	0.012	**0.002**	0.006	0.004	0.008	0.008	**0.002**	**0.002**	0.008	0.004

Table A9. Empirical size of goodness of fit tests based on the D, V, W^2, U^2, A^2, D^*, V^*, W^{2*}, U^{2*}, A^{2*} and r_n statistics when $\mathcal{F}$ is the SGN class and the nominal significance level is 10%.

r&b		$r=-0.7$					$r=-0.2$					$r=0$				
	n	20	50	100	200	500	20	50	100	200	500	20	50	100	200	500
$b=1.5$	D	**0.076**	**0.080**	0.095	0.090	0.090	0.100	0.098	0.084	0.096	**0.080**	**0.072**	0.086	*0.122*	**0.080**	0.092
	V	0.088	**0.070**	0.100	0.100	0.102	0.102	*0.122*	0.100	0.112	0.086	0.096	*0.120*	*0.122*	0.092	0.110
	W^2	**0.080**	**0.064**	0.085	0.096	0.092	0.094	0.104	0.090	0.086	0.086	0.096	0.084	0.082	0.082	0.102
	U^2	0.082	**0.076**	0.105	0.104	0.096	0.102	*0.122*	0.096	0.096	0.090	0.096	0.100	0.096	0.100	0.112
	A^2	0.086	**0.078**	0.098	0.106	0.088	0.102	0.100	0.086	0.106	0.088	0.104	0.104	0.084	**0.080**	0.094
	D^*	0.096	**0.068**	0.087	0.086	0.084	**0.058**	**0.060**	**0.076**	**0.070**	0.118	**0.050**	**0.078**	**0.080**	**0.076**	0.094
	V^*	**0.078**	**0.064**	0.102	0.094	0.094	**0.070**	**0.072**	**0.078**	0.092	0.102	**0.058**	**0.068**	0.090	0.090	0.090
	W^{2*}	0.084	0.084	0.098	0.082	0.082	**0.058**	**0.054**	**0.060**	**0.080**	0.096	**0.052**	**0.052**	**0.076**	**0.054**	**0.078**
	U^{2*}	**0.074**	0.086	0.113	0.094	0.084	**0.062**	**0.062**	**0.066**	**0.072**	0.098	**0.058**	**0.062**	**0.076**	**0.066**	**0.074**
	A^{2*}	**0.080**	**0.074**	0.093	0.092	**0.080**	**0.058**	**0.058**	**0.066**	0.084	0.106	**0.050**	**0.056**	**0.076**	0.082	0.090
	r_n	**0.050**	**0.073**	0.083	0.092	0.088	**0.062**	**0.070**	**0.053**	0.098	**0.072**	**0.078**	**0.037**	**0.067**	**0.064**	**0.074**
$b=2$	D	**0.058**	0.088	**0.074**	**0.076**	0.102	**0.068**	**0.072**	**0.078**	0.084	0.084	**0.072**	0.086	*0.122*	**0.080**	0.092
	V	0.088	**0.080**	0.098	0.102	0.092	**0.070**	0.098	0.086	**0.076**	0.108	0.096	*0.120*	*0.122*	0.092	0.110
	W^2	**0.072**	0.090	**0.076**	**0.048**	0.100	**0.062**	**0.074**	**0.080**	**0.066**	**0.078**	0.096	0.084	0.082	0.082	0.102
	U^2	0.094	0.088	0.082	**0.070**	0.090	**0.072**	0.096	0.104	0.084	0.086	0.096	0.100	0.096	0.100	0.112
	A^2	0.082	0.096	**0.068**	**0.056**	0.090	**0.068**	**0.072**	**0.078**	**0.076**	0.088	0.104	0.104	0.084	**0.080**	0.094
	D^*	0.104	0.094	**0.058**	**0.064**	0.104	**0.062**	**0.038**	**0.058**	**0.050**	0.116	**0.050**	**0.078**	**0.080**	**0.076**	0.094
	V^*	0.092	0.084	0.086	**0.076**	0.094	**0.066**	**0.054**	0.084	**0.070**	0.100	**0.058**	**0.068**	0.090	0.090	0.090
	W^{2*}	**0.072**	**0.080**	**0.054**	**0.052**	**0.076**	**0.056**	**0.034**	**0.050**	**0.064**	0.092	**0.052**	**0.052**	**0.076**	**0.054**	**0.078**
	U^{2*}	0.082	**0.078**	**0.070**	0.082	**0.080**	**0.060**	**0.048**	**0.078**	**0.076**	0.094	**0.058**	**0.062**	**0.076**	**0.066**	**0.074**
	A^{2*}	**0.076**	**0.078**	**0.052**	**0.048**	**0.080**	**0.056**	**0.032**	**0.058**	**0.058**	**0.076**	**0.050**	**0.056**	**0.076**	0.082	0.090
	r_n	0.092	**0.064**	**0.068**	**0.076**	0.090	**0.048**	**0.046**	**0.068**	0.082	0.082	**0.078**	**0.037**	**0.067**	**0.064**	**0.074**
$b=2.5$	D	**0.076**	0.084	0.088	0.086	**0.074**	0.082	0.082	0.086	0.082	0.094	0.094	0.084	0.112	0.082	**0.068**
	V	**0.076**	0.090	0.100	**0.078**	0.086	0.086	**0.074**	0.096	0.088	0.106	0.098	0.104	*0.124*	0.100	0.088
	W^2	0.086	**0.078**	**0.080**	**0.064**	**0.068**	**0.076**	0.084	0.086	**0.066**	0.082	0.104	0.084	0.092	0.096	**0.078**
	U^2	0.092	0.082	0.094	0.084	**0.080**	**0.078**	0.084	0.090	**0.074**	0.102	0.102	0.094	0.116	0.106	**0.080**
	A^2	0.082	0.084	0.082	0.084	**0.076**	**0.066**	0.102	0.102	**0.064**	0.094	0.104	**0.072**	0.100	0.098	**0.080**
	D^*	0.096	**0.080**	0.084	**0.072**	**0.072**	**0.056**	**0.040**	**0.078**	0.090	0.100	**0.080**	0.082	**0.080**	**0.076**	**0.078**
	V^*	**0.074**	0.088	0.088	0.086	**0.078**	**0.070**	**0.042**	**0.070**	0.090	0.100	0.082	**0.072**	0.094	**0.080**	**0.076**
	W^{2*}	0.094	**0.074**	**0.072**	**0.066**	**0.068**	**0.054**	**0.036**	**0.054**	**0.070**	0.094	**0.066**	**0.052**	**0.066**	0.088	**0.062**
	U^{2*}	0.086	**0.080**	**0.078**	0.082	**0.070**	**0.066**	**0.046**	**0.076**	0.082	0.090	**0.072**	**0.066**	**0.074**	0.092	**0.062**
	A^{2*}	0.084	**0.066**	**0.062**	**0.070**	**0.064**	**0.050**	**0.040**	**0.050**	**0.072**	0.086	**0.058**	**0.060**	**0.058**	0.082	**0.052**
	r_n	**0.064**	0.084	**0.072**	0.096	**0.080**	**0.064**	**0.050**	**0.074**	**0.074**	0.086	0.086	**0.068**	**0.074**	**0.072**	**0.068**